

☆ 网课: 西安电子科技大学 — 信号与系统 郭宝龙

☆ 新しい信号処理の教科書 馬場口

工程信号与系统 郭宝龙

Signals and Systems Oppenheim

数字信号处理 John G. Proakis

$x(t)$ 连续时间信号 $x[n]$ 离散时间信号

能量信号: $E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |x[n]|^2$

功率信号: 能量无穷大, 周期性

$$P = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

$$P = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

因果信号: $t < 0$ 值为 0 反因果: $t > 0$, 值为 0

基本信号

单位阶跃信号 单位スツッ信号 $u(t)/\varepsilon(t)$

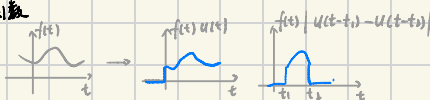
$$u(t) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} r_n(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

1. 表示反因果量信号



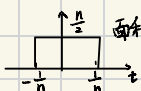
$$f(t) = 2u(t-1) - 3u(t-1) + u(t+2)$$

2. 表示分段常量函数

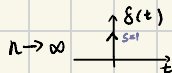


$$3. \int_{-\infty}^t u(\tau) d\tau = tu(t) = \begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$$

单位冲激信号 单位インパルス信号 $\delta(t)$



面积恒为1的函数

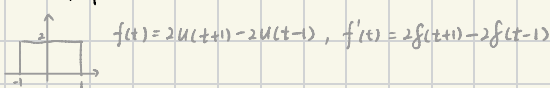


$$\begin{cases} \delta(t) = 0, & t \neq 0 \\ \int_{-\infty}^{+\infty} \delta(t) dt = 1, \end{cases}$$

偶函数

$$\delta(t) = \frac{d}{dt} u(t), \quad u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

描述区间特点



采样性质

$$f(t)\delta(t) = f(0)\delta(t), \quad \int_{-\infty}^{+\infty} f(t)\delta(t) dt = f(0) \int_{-\infty}^{+\infty} \delta(t) dt = f(0)$$

$$f(t)\delta(t-a) = f(a)\delta(t-a), \quad \int_{-\infty}^{+\infty} f(t)\delta(t-a) dt = f(a)$$

冲激偶函数 $f(t)\delta'(t) = f(0)\delta'(t) - f'(0)\delta(t)$

$\delta(t)$ 的导函数

$$[f(t)\delta'(t)]' = f'(t)\delta'(t) + f(t)\delta''(t)$$

$$f(t)\delta''(t) = [f(t)\delta'(t)]' - f'(t)\delta'(t) = f(0)\delta''(t) - f'(0)\delta'(t)$$

$$\int_{-\infty}^{+\infty} f(t)\delta'(t) dt = -f'(0) \Rightarrow \int_{-\infty}^{+\infty} f(t)\delta^{(n)}(t) dt = (-1)^n f^{(n)}(0)$$

$$LHS = f(t)\delta'(t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta'(t)f'(t) dt = 0 - f'(0)$$

尺度变换

$$\delta(at) = \frac{1}{|a|} \delta\left(\frac{t}{a}\right)$$

$$\delta(at) = \frac{1}{|a|} \delta(t)$$

$$\int_{-\infty}^{+\infty} \delta(at)f(t) dt \stackrel{u=at}{=} \int_{-\infty}^{+\infty} \delta(u)f\left(\frac{u}{a}\right) \frac{du}{|a|} = \frac{1}{|a|} f(0), \quad \text{已知 } \int_{-\infty}^{+\infty} \delta(t)f(t) dt = f(0)$$

$$\Rightarrow \delta(at) = \frac{1}{|a|} \delta(t)$$

基本运算

Time reversal 关于y轴翻转

时移 ≥ 0 正向时移(右): $x(t-t_0)$, 延迟 反向时移(左): $x(t+t_0)$, 提前

尺度变换 スケール変換 $f(t) \rightarrow f(ct)$ $c > 1$, 时域上压缩. $0 < c < 1$, 时域上拉伸

系统 输入/激励 (x) → 输出/响应 (y)

连续、离散 取决于输入。

记忆性: 某时刻输出仅取决于该时刻输入 —— 无记忆

因果性: 输出仅取决于当前或过去的输入。 $S.t. t < 0, f(t) = 0$

有界性: 任意时刻 $t, |x(t)| \leq C, |x[n]| \leq C$

稳定性: 输入有界时, 输出有界 —— 有界输入有界输出稳定 BIBO 稳定

时不变 time-invariant system L 表示某系统

$$y(t-t_0) = L(x(t-t_0)) \quad y[n-n_0] = L[x[n-n_0]]$$

eg. 积分器 $y(t) = L[x(t)] = \int_{-\infty}^t x(\tau) d\tau$ 积分器是时不变的

$$y_1(t) = L[x(t-t_0)] = \int_{-\infty}^t x(\tau-t_0) d\tau = \int_{-\infty}^{t-t_0} x(\lambda) d\lambda = y(t-t_0)$$

线性性 $L[ax_1 + bx_2] = aL(x_1) + bL(x_2)$

eg. 乘性 $y(t) = x'(t)$ 非线性

线性时不变系统 LTI

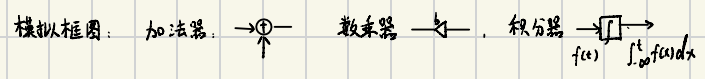
$$f(t) \rightarrow y_{zs}(t), \quad f'(t) \rightarrow y'_{zs}(t), \quad \int_{-\infty}^t f(x) dx \rightarrow \int_{-\infty}^t y_{zs}(x) dx$$

记忆系统的响应 输入 + 过去状态 → 输出

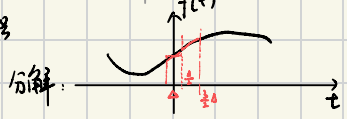
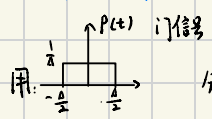
完全响应、零状态响应、受输入响应
不考虑状态

连续系统的描述

微分方程。?



连续信号的时域分解 任意连续函数用单位冲激函数表示



$$\hat{f}(t) = \sum_{n=-\infty}^{\infty} f(nA) \Delta p(t-nA), \quad \lim_{\Delta \rightarrow 0} \hat{f}(t) = f(t) = \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau$$

区间于采样性质 $\int_{-\infty}^{\infty} f(t) \delta(t-a) dt = f(a)$

eg. $y[n] = \sum_{k=-\infty}^n x[k]$ 记忆、因果、非时变, LTI
 $y[n] = x[n] - x[n-1]$ 记忆、因果、时变, LTI

$$\sum x'[n] = x[n-n_0]$$

$$y'[n] = L[x'[n]] = x'[n] - x'[n-1] = y[n-n_0]$$

eg. 单位阶跃函数

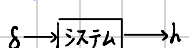
$$u(t) = \int_{-\infty}^{\infty} u(\tau) \delta(t-\tau) d\tau = \int_{-\infty}^{\infty} \delta(t-\tau) d\tau \xrightarrow{p=t-\tau} \int_{-\infty}^{\infty} \delta(t-p) dp$$

$$\text{已知 } u(t) = \int_{-\infty}^t \delta(\tau) d\tau \xleftrightarrow{\text{证明}} = \int_{-\infty}^t \delta(p) dp$$

卷积积分 量纲引入

DEF: $f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \rightarrow f(t) = f_1(t) * f_2(t)$

LTI 的冲激响应 零状态响应



$\delta(t) \rightarrow h(t)$

$\delta(t-\tau) \rightarrow h(t-\tau)$ 时不变

$f(\tau) \delta(t-\tau) \rightarrow f(\tau) h(t-\tau)$ 齐次性

$\int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau \rightarrow \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau$

$f(t) \rightarrow y_{zs}(t)$

记 $y_{zs}(t) = \int_{-\infty}^{\infty} f(\tau) h(t-\tau) d\tau = f(t) * h(t)$

引入卷积积分方便书写

与冲激响应作卷积: LTI 零状态响应

时域分解可记为 $f(t) = \int_{-\infty}^{\infty} f(\tau) \delta(t-\tau) d\tau = f(t) * \delta(t)$ 与冲激函数作卷积 $\rightarrow$ 时域分解

交换、分配、结合 $\delta * x = x * \delta = x$ "单位"

LTI 阶跃响应

$u \rightarrow s$

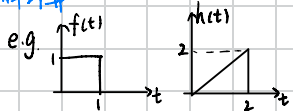
$t-\tau > 0, \tau < t$

$s(t) = u(t) * h(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau = \int_{-\infty}^t h(\tau) d\tau$

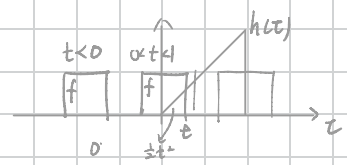
即有 $h(t) = \frac{d s(t)}{dt}$, $s(t) = \int_{-\infty}^t h(\tau) d\tau$ 即符合 LTI 系统的微积分特性。

e.g. $\lambda h(x(t)) = e^{-at} u(t), a > 0, h(t) = u(t)$ $\Leftrightarrow t > 0$ 有值
 出力 $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau = \int_0^t x(\tau) d\tau = \frac{1}{a}(1-e^{-at})$ $t > 0$
 对所有 $t, y(t) = \frac{1}{a}(1-e^{-at}) u(t)$

图解计算



$h(t) * f(t) = \int_{-\infty}^{\infty} h(\tau) f(t-\tau) d\tau$



门函数与自己卷积得到一个三角形



奇异函数卷积特性

$\delta(t), u(t)$

$f(t) * \delta(t) = f(t), f(t) * \delta(t-t_0) = f(t-t_0)$

$\tau=0 \quad t=\tau$
 $f'(\tau) \delta(t-\tau) = f'(t)$

冲激偶: $f(t) * \delta'(t) = f'(t), f(t) * \delta^{(n)}(t) = f^{(n)}(t)$

LHS = $\int_{-\infty}^{\infty} f(\tau) \delta'(t-\tau) d\tau = - \int_{-\infty}^{\infty} f'(\tau) \delta(t-\tau) d\tau = f'(t)$

$f(t) * u(t) = \int_{-\infty}^t f(\tau) d\tau, u(t) * u(t) = \int_0^t 1 d\tau = t u(t)$ 相当于 $t > 0$

卷积时移

用换元易得

$f(t) = f_1(t) * f_2(t)$, 有 $f_1(t-t_1) * f_2(t-t_2) = f_1(t-t_1-t_2) * f_2(t) = f_1(t) * f_2(t-t_1-t_2) = f(t-t_1-t_2)$

e.g. $f_1(t)$ 如图, $f_2(t) = e^{-t} u(t)$, 求 $f_1 * f_2$

$f_1(t) = u(t) - u(t-2)$



$$f_1(t) = f_1 * f_2 = u(t) * f_2 - u(t-1) * f_2$$

$$u(t) * f_2 = \int_{-\infty}^{\infty} e^{-\tau} u(t-\tau) d\tau = \int_0^t e^{-\tau} d\tau = (-e^{-t} - 1) u(t)$$

$$\Rightarrow u(t-1) * f_2 = (-e^{-t+1} - 1) u(t-1) \quad \square$$

卷积微积分性质

$$1. \frac{d}{dt^n} [f_1(t) * f_2(t)] = \frac{d^n}{dt^n} f_1(t) * f_2(t) = \frac{d^n}{dt^n} f_2(t) * f_1(t) \quad \text{LHS} = \frac{d^n}{dt^n} (f_1(t) * f_2(t)) = f_1^{(n)}(t) * f_2(t) = f_1(t) * f_2^{(n)}(t)$$

$$2. \int_{-\infty}^t [f_1(\tau) * f_2(\tau)] d\tau = [\int_{-\infty}^t f_1(\tau) d\tau] * f_2(t) = \dots$$

$$3. f_1(-\infty) = 0 \text{ 或 } f_2^{(-1)}(0) = 0, \quad f_1 * f_2 = f_1' * f_2^{(-1)}$$

与f(t)卷积: $\delta(t) h(t) \quad \delta'(t) u(t)$

$$f(t) y_{zs}(t) \quad f'(t) \int_{-\infty}^t f(\tau) d\tau$$

利用梳状函数产生周期信号

$$y(t) = f(t) * \delta(t-t_0) = f(t-t_0)$$

梳状函数 $\delta_T(t)$ 是以T为周期的冲激序列

$$\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t-mT)$$

$$f(t) * \delta_T(t) = \sum_{m=-\infty}^{\infty} f(t-mT)$$

分配, 采样

要求f的定义域 < T, 否则发生混叠



卷积计算公式

$$k * f(t) = \int_{-\infty}^{\infty} f(\tau) d\tau$$

冲激函数

$$f(t) * \delta(t) = f(t), \quad f(t) * \delta'(t) = f'(t) * \delta(t) = f'(t)$$

$$f(t) * u(t) = f(t) * \int_{-\infty}^t \delta(\tau) d\tau = \int_{-\infty}^t f(\tau) d\tau \quad (t < 0 \text{ 为 } 0)$$

$$u(t) * u(t) = t u(t)$$

$$e^{-a_1 t} u(t) * e^{-a_2 t} u(t) = t e^{-a_1 t} u(t)$$

$$e^{-a_1 t} u(t) * e^{-a_2 t} u(t) = \frac{e^{-a_1 t} - e^{-a_2 t}}{a_1 - a_2} u(t) \quad (a_1 \neq a_2)$$

$$f(t) * \delta_T(t) = f(t) * \sum_{m=-\infty}^{\infty} \delta(t-mT) = \sum_{m=-\infty}^{\infty} f(t-mT)$$

相关函数

互相关函数 2个信号在不同延迟下的相似程度

$$R_{12}(T) = \int_{-\infty}^{\infty} f_1(t) f_2(t-T) dt = \int_{-\infty}^{\infty} f_1(t+T) f_2(t) dt \quad f_1 \text{ 领先 } f_2$$

$$R_{11}(T) = R_{11}(-T)$$

当 f_1, f_2 都为偶函数时, $R_{12} = f_1 * f_2$

自相关函数 $R(t) = \int_{-\infty}^{\infty} f(t) f(t-\tau) d\tau = R(-\tau)$

离散系统 $f[n]/f[k]$

单位脉冲序列 $\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$ 偶函数

$\sum_{n=-\infty}^{\infty} \delta[n] = 1$ 信号 $f[n]\delta[n] = f[0]\delta[n], f[n]\delta[n-n_0] = f[n_0]\delta[n-n_0]$ $\sum f[n]\delta[n] = f[0], \sum f[n]\delta[n-n_0] = f[n_0]$

叠分: $\sum_{i=-\infty}^k \delta[i] = \begin{cases} 0, & k < 0 \\ 1, & k \geq 0 \end{cases}$

单位阶跃序列 $u[k]/u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$\delta[n] = u[n] - u[n-1]$ 一次差分

$u[n] = \sum_{i=0}^n \delta[i]$

差分方程

$\Delta x[n] \stackrel{\text{def}}{=} x[n] - x[n-1]$ 信号-延迟信号

线性性质。

$\Delta^2 x[n] = \Delta[\Delta x[n]] = \Delta x[n] - \Delta x[n-1]$ 高阶差分

差分方程求解

模拟框图。

时序分解 $f[n] = \sum_{i=-\infty}^n f[i]\delta[n-i]$ 卷积和

卷积和

$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ $\rightarrow f_1[n] \cdot u[n]$ (仅保留0的部分)

上下限: $f = f_1 * f_2$, 有 ① f_1 为因果, 则 $f[n] = \sum_{i=0}^n f_1[i]f_2[n-i]$ ② f_2 为因果, $f[n] = \sum_{i=0}^n f_1[i]f_2[n-i]$

③ 都为因果 $f[n] = \sum_{i=0}^n f_1[i]f_2[n-i]$

零状态响应 y_{zs}

已知 $\delta[n] \rightarrow h[n] \Rightarrow f[n] \rightarrow f[n] * h[n] = y_{zs}[n]$

$f[i]\delta[k-i] \rightarrow f[i]h[k-i]$

$\sum f[i]\delta[k-i] \rightarrow \sum f[i]h[k-i]$

e.g. ?

卷积和不进位乘法 一种卷积和计算方法

$f[k] = f_1[k] * f_2[k] = \sum_{i=-\infty}^k f_1[i]f_2[k-i]$ 所有序号之和为k的乘积之和

e.g. $f_1[k] = \{0, 2, 1, 5, 0\}$, $f_2[k] = \{0, 3, 4, 0, 6, 0\}$

取非0, 不进位

$\begin{matrix} & 3 & 4 & 0 & 6 & f_2 \\ \times & 2 & 1 & 5 & f_1 \\ \hline 15 & 20 & 0 & 30 \end{matrix}$

e.g. $k=4, f[4] = f_1[0]f_2[4] + f_1[1]f_2[3] + f_1[2]f_2[2] + f_1[3]f_2[1] + f_1[4]f_2[0]$
 $= 12 + 20 = 32$

$\begin{matrix} f_1[n]f_2[n] & 0 & 11 & 19 & 32 & 6 & 30 & 6 \end{matrix}$

卷积公式

$$f(k) * \delta(k) = f(k), f(k) * \delta(k-k_0) = f(k-k_0), \delta(k) * \delta(k) = \delta(k)$$

$$f(k) * \varepsilon(k) = \sum_{i=-\infty}^k f(i)$$

$$f_1(k-k_1) * f_2(k-k_2) = f_1(k-k_1) * f_2(k-k_1) = f_1(k) * f_2(k-k_1-k_2) = \checkmark$$

卷积和时移

信号正交分解与傅氏级数

$$\vec{v} = C_1 \vec{v}_1 + C_2 \vec{v}_2, \vec{v} \cdot \vec{v} = C_1 \vec{v}_1 \cdot \vec{v}_1 + C_2 \vec{v}_2 \cdot \vec{v}_2 = |\vec{v}|^2 \Rightarrow C_1 = \frac{\vec{v} \cdot \vec{v}_1}{|\vec{v}_1|^2}$$

$$n \text{ 维: } \vec{v} = C_1 \vec{v}_1 + \dots + C_n \vec{v}_n, \vec{v}_1 \cdot \vec{v}_j = 0 \Rightarrow C_r = \frac{\vec{v} \cdot \vec{v}_r}{\vec{v}_r \cdot \vec{v}_r}$$

$$\text{连续信号 } \langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) y^*(t) dt, y^*(t) \text{ 是 } y(t) \text{ 的复共轭 } (y(t) \text{ 为实, } y^*(t) = y(t))$$

$$\text{离散信号 } \langle x[n], y[n] \rangle = \sum x[n] y^*[n]$$

内积 = 0, 正交

$$\text{范数: } \|x(t)\| = \sqrt{\langle x(t), x(t) \rangle}$$

模

$$\text{正交函数集 } \langle f_1(t), \dots, f_n(t) \rangle \text{ s.t. } \langle f_i, f_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \rightarrow \text{标准正交函数集}$$

完备性

$$\text{任意 } x(t), t \in [t_1, t_2], \text{ s.t. } x(t) = \sum_{i=0}^{\infty} C_i f_i(t), C_i = \frac{\int_{t_1}^{t_2} x(t) f_i^*(t) dt}{\int_{t_1}^{t_2} f_i(t) f_i^*(t) dt}$$

正交函数集线性组合

$$\text{常见完备正交集 } t \in (t_0, t_0+T), T = 2\pi/\Omega$$

$$\text{三角函数集: } \{1, \cos(n\Omega t), \sin(n\Omega t), n=1, 2, \dots\}$$

$$\text{虚指数集: } \{e^{jn\Omega t}, n=0, \pm 1, \pm 2, \dots\}$$

$$\text{任意信号正交分解 } f(t) = C_1 \varphi_1(t) + C_2 \varphi_2(t) + \dots = \sum_{i=1}^{\infty} C_i \varphi_i(t) \quad \text{广义傅里叶级数}$$

$$C_i = \frac{\langle f, \varphi_i \rangle}{\langle \varphi_i, \varphi_i \rangle}$$

周期信号分解

$$\text{泰勒级数 } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$\text{三角函数集: } 1, \cos x, \sin x, \dots \quad [-\pi, \pi]$$

$$\langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx dx = 0, \quad \langle 1, \sin nx \rangle = 0$$

$$\langle \cos nx, \sin nx \rangle = \int_{-\pi}^{\pi} \cos nx \sin nx dx = 0 \quad \text{积化和差}$$

$$\text{周期函数内积: 定义在 } [-\pi, \pi], \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) g(x) dx, \|f\| = \sqrt{\langle f, f \rangle}, \quad \|\cos nx\| = \|\sin nx\| = 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{周期为 } 2\pi$$

要求 f(x) 有收敛性、收敛单调

$$a_n = \langle f(x), \cos nx \rangle, \quad b_n = \langle f(x), \sin nx \rangle$$

$$S(x) = \begin{cases} f(x) & x \text{ 连续点} \\ \frac{f(x+0) + f(x-0)}{2} & x \text{ 为间断点} \end{cases}$$

$$f(x) \text{ 为偶: } b_n = 0, \quad f(x) \text{ 为奇: } a_n = 0$$

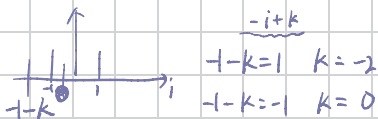
$$\text{任意: 周期为 } T \quad [-T/2, T/2] \quad \langle f(t), g(t) \rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(t) g(t) dt$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega t + \sum_{n=1}^{\infty} b_n \sin n\Omega t \quad \left. \begin{array}{l} \text{直流分量: } a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt \\ a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\Omega t dt \\ b_n = \dots \end{array} \right\}$$



Week 4

$$1. f_1(k) * f_2(k) = \sum_{n=-\infty}^{\infty} f_1[n] f_2[k-n] = \begin{cases} 2, & k = -2/0 \\ 0, & \text{其他} \end{cases}$$



$$\begin{aligned} n=0 & \rightarrow 1 \\ \cos(-n\pi) &= (-1)^{n+1} \\ n=0 & \rightarrow 1 \\ \cos n\pi &= (-1)^n \end{aligned}$$

2. $T=2\pi$ 奇函数.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad a_n = 0$$

$$\begin{aligned} b_n &= \langle f(x), \sin nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin nx \cdot f(x) dx = - \int_{-\pi}^0 \sin nx dx + \int_0^{\pi} \sin nx dx = \frac{1}{n\pi} \cos nx \Big|_{-\pi}^0 - \frac{1}{n\pi} \cos nx \Big|_0^{\pi} \\ &= \frac{1}{n\pi} [1 - (-1)^{n+1} - (-1)^n + 1] \\ &= \frac{1}{n\pi} \cdot 2 \end{aligned}$$

$$S(-\pi) = 0 = S(\pi) = S(0)$$

$$S(x) = \begin{cases} 0, & \pm n\pi \\ f(x), & \text{其他} \end{cases}$$

3. $\frac{1}{x}$ 偶, $b_n = 0$

$$\begin{aligned} a_n &= \langle f(x), \cos nx \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{1}{n\pi} \int_{-\pi}^{\pi} x^2 d \sin nx \\ &= \frac{1}{n\pi} \left[x^2 \sin nx \Big|_{-\pi}^{\pi} - 2 \int_{-\pi}^{\pi} x \sin nx dx \right] \end{aligned}$$

$$\begin{aligned} a_0 &= \langle f(x), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx \\ &= \frac{2}{\pi} \int_0^{\pi} d \frac{1}{3} x^3 = \frac{2}{3\pi} \pi^3 = \frac{2}{3} \pi^2 \end{aligned}$$

$$\begin{aligned} &= \frac{2}{n\pi} \int_{-\pi}^{\pi} x \sin nx dx = \frac{2}{n^2\pi} \int_{-\pi}^{\pi} x d \cos nx \\ &= \frac{2}{n^2\pi} \left[x \cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \cos nx dx \right] \\ &= \frac{2}{n^2\pi} 2\pi \cos n\pi = \frac{4}{n^2} (-1)^n \end{aligned}$$

$$f(x) = \frac{x^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$$

4. 偶, $b_n = 0$, $T = T$

$$\frac{1}{2} a_0 = \frac{1}{2} \langle f(t), 1 \rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{2}{T} \int_0^{T/2} A \sin \omega t dt = \frac{-2A}{\omega T} \cos \omega t \Big|_0^{T/2} = \frac{-2A}{\omega T} (\cos \frac{\omega T}{2} - 1) = \frac{4A}{\omega T}$$

$$a_n = \langle f(t), \cos n \frac{2\pi}{T} t \rangle$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n \frac{2\pi}{T} t dt$$

$$= \frac{4}{T} \int_0^{T/2} A \sin \omega t \cos n \omega t dt \quad \omega t = \frac{2\pi}{T} t = k \quad t = T/2, k = \pi$$

$$= \frac{2A}{T} \int_{-\pi}^{\pi} \sin k \cos n k dk$$

$$a_1 = \frac{4A}{T} \cdot \pi, \quad n \geq 2, a_n = 0$$

$$\begin{aligned} f(t) &= \frac{4A}{\omega T} + \frac{2A}{T} \cdot \pi \cdot \cos \frac{2\pi}{T} t \\ &= \frac{2A}{\pi} + \frac{2A\pi}{T} \cos \frac{2\pi}{T} t \end{aligned}$$

