

$$\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}, \quad \frac{d}{dx} x^n = nx^{n-1}, \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1, \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\theta} = 0$$

连续: $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ 极限存在且等于函数值。可导一定连续, 连续不一定可导。

$f(x)$ 是偶函数 $\rightarrow f'(x)$ 是奇函数, vice versa

$$\frac{d}{dx} \sin x = \cos x, \quad \frac{d}{dx} \cos x = -\sin x, \quad \frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$$

$$(u+v)' = u' + v', \quad (uv)' = u'v + uv', \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$f^{(n)}(x) = D^n f = \frac{d^n f}{dx^n}, \quad \text{exp. } D^n x^n = n!$$

implicit differentiation: 两边求导

$$y = f(x), \quad x = g(y) \Rightarrow x = g(y) = f^{-1}(y), \quad \text{exp. } \frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx} e^x = e^x, \quad \frac{d}{dx} \ln x = \frac{1}{x}, \quad \frac{d}{dx} a^x = a^x \cdot \ln a, \quad \frac{d}{dx} x^x = x^x (\ln x + 1)$$

$$f' = f(\ln f)'$$

$$\lim_{x \rightarrow \infty} \ln(1 + \frac{1}{x})^x = e \quad \frac{f'(t)}{f(t)} = \frac{d}{dt} (\ln f(t))$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \cosh^2(x) - \sinh^2(x) = 1 \quad \text{双曲函数}$$

$$\frac{d}{dx} \sinh(x) = \cosh(x), \quad \frac{d}{dx} \cosh(x) = \sinh(x)$$

线性近似 $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$

exp. Baspoint $x_0 = 0$: $\sin x \approx x, \cos x \approx 1, e^x \approx 1 + x, \ln(1+x) \approx x, (1+x)^r \approx 1 + rx$

$$\text{for } f(x) = \frac{e^{-2x}}{\sqrt{1+x}}: \begin{aligned} e^{-2x} &\approx 1 - 2x \\ \sqrt{1+x} &\approx 1 + \frac{1}{2}x \end{aligned} \rightarrow f(x) \approx (1 - 2x)(1 + \frac{1}{2}x)^{-1} \approx (1 - 2x)(1 - \frac{1}{2}x) \approx 1 - \frac{5}{2}x \quad \text{drop } x^2 \quad (x \ll 1)$$

$$\text{So } f(0) = 1, \quad f'(0) = -\frac{5}{2}$$

不指定教材

可参考的书目:

- ① 大学院への数学 (基础难度)
 - ② 寺田の演習微積分 (中等难度)
 - ③ 马场敬之: 微積分/ベクトル解析 キャンパス・ゼミ (基础难度)
 - ④ 小黄书上册 (高难度, 面向东/京两校)
 - ⑤ MIT的微积分课程配套
 - ⑥ 同济的高等数学 (快速查阅手册)
 - ⑦ 各类国内考研辅导书 (导数、积分相关的填空和小计算题, 日常练手)
 - ⑧ ***部分数学分析教材的特定章节 (面向东大)
 - (1) 谢惠民: 数学分析习题课讲义
 - (2) 裴礼文: 数学分析中的典型问题与方法 } 难
 - (3) 复旦/华师大版的数分教材
- (**以上只看计算相关的章节)

极限理论

1. 数列极限 $\lim_{n \rightarrow \infty} a_n = \alpha \iff \forall \varepsilon > 0, \exists N > 0, \text{ s.t. } \forall n > N, |a_n - \alpha| < \varepsilon$

否命题: $\{a_n\}$ 极限不是 α / 不存在

$$\exists \varepsilon > 0, \forall N > 0, \exists N_0 > N, |a_{N_0} - \alpha| > \varepsilon$$

证: $\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$

$$n > \log_2 \frac{1}{\varepsilon}, \frac{1}{2^n} < \varepsilon \Rightarrow N = \lceil \log_2 \frac{1}{\varepsilon} \rceil + 1, \forall n > N, |\frac{1}{2^n} - 0| < \varepsilon$$

常用极限

分式只看最高次项系数

$$\lim_{n \rightarrow \infty} \frac{3n^2 + 5n + 1}{n^2 + n + 1} = 3$$

$$\lim_{n \rightarrow \infty} \frac{7n^3}{\sqrt{n^3 + n}} + n = 7$$

$$\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e = 2.718... \quad 1^{\infty}$$

2. Cauchy 收敛 $\forall \varepsilon > 0, \exists N > 0, \text{ s.t. } \forall m, n > N, |a_m - a_n| < \varepsilon \iff \lim_{n \rightarrow \infty} a_n \exists$ 极限存在/数列收敛

证收敛 $a_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \dots + \frac{\sin n}{2^n}$ 不妨 $m < n$

$$\forall \varepsilon > 0, |a_m - a_n| \leq \left| \frac{\sin(m+1)}{2^{m+1}} \right| + \dots + \left| \frac{\sin n}{2^n} \right| \leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots + \frac{1}{2^n} = \frac{1}{2^{m+1}} (1 - \frac{1}{2^{n-m}}) = \frac{1}{2^n} (1 - \frac{1}{2^{n-m}}) < \frac{1}{2^n} < \varepsilon \Rightarrow m > \log_2 \frac{1}{\varepsilon}$$

3. 单调有界收敛 (归纳) $n \rightarrow +\infty, \{a_n\} \nearrow$ 有上界 / $\searrow$ 有下界, 收敛

证收敛: $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2a_n} \parallel a_1 = \sqrt{2}, a_{n+1} = \sqrt{2 + a_n}$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{2}}{\sqrt{a_n}}$$

$$a_1 = \sqrt{2} < 2, \text{ 若 } a_n < 2, a_{n+1} = \sqrt{2a_n} < 2 \quad \square \quad \text{故 } \{a_n\} < 2 \text{ 且单调递增 } \checkmark$$

求极限: 假设 $\lim_{n \rightarrow \infty} a_n = a, a_{n+1} = \sqrt{2a_n} \xrightarrow{n \rightarrow \infty} a = \sqrt{2a} \rightarrow a = 2$

4. 夹逼定理 $a_n \leq b_n \leq c_n, \text{ if: } a_n \rightarrow k, c_n \rightarrow k, \text{ then: } b_n \rightarrow k$

$$\lim_{n \rightarrow \infty} (\frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{n+n}}) = \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{n!} = 0 \quad (n^n \gg n! \gg a_n)$$

$$\text{令 } k > a, \frac{a^n}{n!} = \frac{a}{1} \cdot \frac{a}{2} \cdot \dots \cdot \frac{a}{k} \cdot \frac{a}{k+1} \cdot \dots \cdot \frac{a}{n} \leq \frac{a^k}{k!} \left(\frac{a}{k}\right)^{n-k} = \frac{a^k}{k!} \left(\frac{a}{k}\right)^n \rightarrow 0$$

$$a_n \leq \frac{1}{\sqrt{n+1}} + \dots + \frac{1}{\sqrt{n+n}} = \frac{n}{\sqrt{n+1}} \rightarrow 1$$

$$a_n \geq \frac{1}{\sqrt{n+n}} \rightarrow 1$$

5. 函数极限

$$1) x \rightarrow +\infty \quad \lim_{x \rightarrow +\infty} f(x) = A \leftrightarrow \forall \varepsilon > 0, \exists M > 0, \text{ s.t. } \forall x > M, |f(x) - A| < \varepsilon$$

$$2) x \rightarrow \infty, -\infty$$

$$3) x \rightarrow x_0 \quad \text{if } f(x) \text{ 在 } U^\circ(x_0, \delta) \text{ 有定义 then: } \forall \varepsilon > 0, \exists \delta > 0 (\delta < \delta'), \text{ s.t. } \forall 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon$$

$$\lim_{x \rightarrow x_0} f(x) = A \longleftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) = A$$

6. 连续性

Def: $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

间断点:

① 可去: 左=右, $f(x_0) \neq \lim_{x \rightarrow x_0} f(x)$

② 跳跃: 左≠右

③ 无穷: 左/右 $\rightarrow \infty$

连续性质: ① 闭区间有界

② 介值定理 $f(a) < f(b), \forall A \in (f(a), f(b)), \exists x \in (a, b), f(x) = A$

★ $f(x)$ 连续, $\lim_{n \rightarrow \infty} f(x_n) = 0, \beta > 1, J_m(\beta) = \frac{1}{\beta} [f(m\beta) + f(\frac{\beta}{m})]$ 数列收敛?

$$\forall m, n, |J_m(\beta) - J_n(\beta)| = \frac{1}{\beta} [f(m\beta) - f(n\beta) + f(\frac{\beta}{m}) - f(\frac{\beta}{n})]$$

$$\leq \frac{1}{\beta} |f(m\beta) - f(n\beta)| + \frac{1}{\beta} |f(\frac{\beta}{m}) - f(\frac{\beta}{n})|$$

① 由 $f(x) \rightarrow 0, \forall \varepsilon > 0, \exists N' > 0, \text{ s.t. } \forall m\beta, n\beta > N', |f(m\beta) - f(n\beta)| < \frac{1}{2}\beta\varepsilon$ ← 任取常数, 利用柯西收敛准则

② 由连续: $\lim_{x \rightarrow 0} f(x) = f(0), f(\frac{\beta}{n}) = b_n, \{b_n\} \rightarrow f(0)$ 同理 $\{a_n\}$

$$\exists N'' > 0: \forall m, n > N'', |f(\frac{\beta}{m}) - f(\frac{\beta}{n})| < \frac{1}{2}\beta\varepsilon$$

由 ①②. $N = \lceil \max\{\frac{N'}{\beta}, N''\} \rceil + 1, \forall m, n > N, |J_m - J_n| \leq \varepsilon$

7. 极限运算: 不回避拆分

$$\begin{matrix} f(x) \rightarrow A \\ g(x) \rightarrow B \end{matrix} \Rightarrow \begin{matrix} f(x) + g(x) \rightarrow A+B, \\ f(x) \cdot g(x) \rightarrow A \cdot B, \\ f(x)/g(x) \rightarrow A/B \end{matrix}$$

多项式极限: 比较最高次项系数

不定式: $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 1^\infty, 0 \cdot \infty \dots$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1, \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e, \lim_{x \rightarrow 0} x^x = 1, \lim_{x \rightarrow +\infty} x^{\frac{1}{x}} = 1, \lim_{x \rightarrow 0^+} x \ln x = 0, \lim_{x \rightarrow +\infty} \frac{\ln x}{x} = 0, \lim_{x \rightarrow +\infty} \frac{x^a}{e^x} = 0$$

$\ln x^a \rightarrow \frac{a \ln x}{x} \rightarrow \frac{-\ln x}{\frac{1}{x}}$

$x^n \sim n! > n! > e^n \sim e^x > x^a > \ln x$ n 指数 (数列极限)

等价无穷小：乘除替换

$$x \rightarrow 0: x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim e^x - 1 \sim \ln(1+x)$$

$$\ln(1+x) \sim e^x - 1$$

$$1 - \cos x \sim \frac{1}{2}x^2$$

$$(1+x)^a - 1 \sim ax$$

$$a^x - 1 \sim x \ln a$$

$$x - \ln(1+x) \sim \frac{1}{2}x^2$$

$$\tan x - \sin x \sim \frac{1}{2}x^3, x - \sin x \sim \frac{1}{6}x^3, \tan x - x \sim \frac{1}{3}x^3, x - \arctan x \sim \frac{1}{3}x^3$$

$$\lim_{x \rightarrow 0} \frac{x - \arctan x}{x^k} = c \neq 0 \text{ 求 } k, c$$

$$\lim_{x \rightarrow 0} x(\sqrt{1+x^2} - x) \rightarrow \frac{x(1+x^2-x^2)}{\sqrt{1+x^2}+x} = 0$$

$$\rightarrow \frac{\frac{1}{2}x^3}{x^k} \rightarrow c \neq 0 \Rightarrow \begin{cases} k=3 \\ c=\frac{1}{2} \end{cases}$$

洛必达: $\frac{0}{0}$

$$1^\infty \text{ 问题: } \lim A^B = e^{\lim (A-1)B} \quad (A \rightarrow 1, B \rightarrow \infty)$$

$$\lim_{x \rightarrow 0} (1 + \sin 2x)^{\frac{1}{\sin 2x}} \rightarrow (1 + \sin 2x)^{\frac{1}{\sin 2x} \cdot \sin 2x \cdot \frac{1}{x}} \rightarrow e^{\frac{\sin 2x}{x}} \rightarrow e^2$$

$$\lim_{x \rightarrow \infty} (x \arctan \frac{1}{x})^{\frac{1}{x}} \xrightarrow{x=\frac{1}{t}} \left(\frac{1}{t} \arctan t\right)^{\frac{1}{\frac{1}{t}}} \rightarrow e^{(\arctan t - t) \cdot \frac{1}{t^2}} \rightarrow e^{-\frac{1}{3}}$$

无穷小·有界量 $\lim_{x \rightarrow \infty} e^{-x} \sin x = 0$ e.g. $\lim_{x \rightarrow \infty} \frac{x^2+x+1}{2^x+x^2} (\sin x + \cos x) = 0$, 由 $\frac{x^2+x+1}{2^x+x^2} \rightarrow 0$, $\sin x + \cos x < 2$ 得

渐近线

① 垂直: $f(x_0) \rightarrow \infty$

② 水平: $\lim_{x \rightarrow \infty} f(x) \rightarrow A, \lim_{x \rightarrow -\infty} f(x) \rightarrow B$

③ 斜: $k = \lim_{x \rightarrow \infty} \frac{f(x)}{x}, b = \lim_{x \rightarrow \infty} f(x) - kx$

e.g. 1. $\lim_{n \rightarrow \infty} (1 + \frac{1}{n^2})(1 + \frac{1}{n^2}) \dots (1 + \frac{1}{n^2}) =$

令 $X_n = (1 + \frac{1}{n^2}) \dots (1 + \frac{1}{n^2})$

$S_n = \ln X_n = \sum_{i=1}^n \ln(1 + \frac{1}{n^2})$ 由 $\frac{x}{1+x} < \ln(1+x) < x$

$\therefore S_n < \sum_{i=1}^n \frac{1}{n^2} \rightarrow \frac{1}{2} (n \rightarrow \infty), S_n > \sum_{i=1}^n \frac{1/n^2}{1+1/n^2} = \sum_{i=1}^n \frac{i}{n^2+i} > \frac{1+\dots+n}{n^2+n} \rightarrow \frac{1}{2}$

综上: $X_n \rightarrow e^{\frac{1}{2}}$

2. 证明 $\sum_{n=1}^{\infty} \frac{1}{n^2} + \frac{1}{n^3} + \dots + \frac{1}{n^p}$ 数列收敛

令 $m = n+p$ ($p \in \mathbb{Z}^+$)

$$\begin{aligned} |a_m - a_n| &= |a_{n+p} - a_n| = \left| \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(n+p)^2} \right| \\ &\leq \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p-1)(n+p)} \text{ 裂项} \\ &= \frac{1}{n} - \frac{1}{n+p} < \frac{1}{n} \end{aligned}$$

if $|a_m - a_n| < \varepsilon$, then: $\frac{1}{n} < \varepsilon$ 即 $n > \frac{1}{\varepsilon}$

令 $N = \lceil \frac{1}{\varepsilon} \rceil + 1$, $\forall \varepsilon > 0$, s.t. $\forall m, n > N, |a_m - a_n| < \varepsilon$ $\square$

3. 证明 $\lim_{x \rightarrow 0} \sin x = 0$

$\forall \varepsilon > 0$, 找 δ , s.t. $0 < |x-0| < \delta, |\sin x - 0| < \varepsilon$

$\therefore x < \varepsilon, |\sin x| < |x| < \varepsilon$

$\therefore \delta = \varepsilon, \forall \varepsilon > 0$, s.t. $0 < |x-0| < \delta, |\sin x - 0| < \varepsilon$ $\square$

4. 证明 $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$

要找 $N > 0$, 使 $\forall \varepsilon > 0$, s.t. $n > N$, $|\frac{1}{n!} - 0| < \varepsilon$, 即 $\lim_{n \rightarrow \infty} \frac{1}{n!} = 0$

$\therefore \lim_{n \rightarrow \infty} (\frac{1}{n!})^n = 0$ ($n! > a^n$), 即 $\forall \varepsilon' > 0$, $\exists N' > 0$, s.t. $n > N'$, $(\frac{1}{n!})^n < \varepsilon'$

不妨取 $\varepsilon' = 1$, 有 $\exists N' > 0$, s.t. $n > N'$, $(\frac{1}{n!})^n < 1$ //

$\therefore \forall \varepsilon > 0$, 取 $N = N'$, $\forall n > N$, $(\frac{1}{n!})^n < 1$, 即 $|\frac{1}{n!} - 0| < \varepsilon$ $\square$

导数与微分

1. 导数 $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0)$

不可导. e.g. $f(x) = |x|$. $\lim_{\Delta x > 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = 1$, $\lim_{\Delta x < 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = -1$ 极限不存在
(单侧导数)

$f'(x)$: 定义域上每一个点可导, 端点单侧可导. $f'(x) = \frac{dy}{dx} = \frac{d}{dx} f(x)$

e.g. $\frac{d \sin x}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} =$
积化和差...

(1) $(C)' = 0$,	(2) $(x^n)' = nx^{n-1}$,
(3) $(\sin x)' = \cos x$,	(4) $(\cos x)' = -\sin x$,
(5) $(\tan x)' = \sec^2 x$,	(6) $(\cot x)' = -\csc^2 x$,
(7) $(\sec x)' = \sec x \tan x$,	(8) $(\csc x)' = -\csc x \cot x$,
(9) $(a^x)' = a^x \ln a$ ($a > 0, a \neq 1$),	(10) $(e^x)' = e^x$,
(11) $(\log_a x)' = \frac{1}{x \ln a}$ ($a > 0, a \neq 1$),	(12) $(\ln x)' = \frac{1}{x}$,
(13) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$,	(14) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$,
(15) $(\arctan x)' = \frac{1}{1+x^2}$,	(16) $(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$,

2. $(u+v)' = u' + v'$, $[u(x)v(x)]' = u'v + uv'$, $[\frac{u}{v}]' = \frac{u'v - uv'}{v^2}$
链式法则: $y = u(v(w(x)))$, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$

3. 反函数

$y = f(x)$ 的反函数 $x = \varphi(y)$. 若 $\varphi(y)$ 在 y_0 的某邻域连续且严格单调, $\varphi'(y_0) \neq 0$

则 $f(x)$ 在 $x_0 (x_0 = \varphi(y_0))$ 可导: $f'(x_0) = \frac{1}{\varphi'(y_0)}$

e.g. $y = \cos x$, $x = \arccos y$ $y \in (-1, 1)$, $x \in [0, \pi]$

$$1 = \frac{dy}{dx} \arccos y \cdot \frac{dy}{dx}$$

$$\frac{d}{dy} \arccos y = -\frac{1}{\sin x} = \frac{1}{-\sqrt{1-\cos^2 x}}$$

$$= -\frac{1}{\sqrt{1-y^2}}$$

微分表示: $y = f(x)$, $x = g(y) = f^{-1}(x)$

$$\Rightarrow f^{-1}(y) = x$$

$$\frac{d}{dx} f^{-1}(y) = 1$$

$$\frac{d}{dy} f^{-1}(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{dy/dx} \quad \square$$

4. 参数曲线

$$\begin{cases} x = u(t) \\ y = v(t) \end{cases}, \alpha \leq t \leq \beta \longrightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v'(t)}{u'(t)}$$

5. 隐函数 (参数曲线未知)

$x^2 + y^2 = 1$, $2x + 2y \cdot y' = 0 \rightarrow y' = -\frac{x}{y}$

6. 对数求导

$$u(x)^{v(x)} = y$$

$$\ln y = v(x) \ln u(x)$$

$$\frac{1}{y} \cdot y' = v'(x) \ln u(x) + v(x) \cdot \frac{1}{u(x)} u'(x)$$

e.g. $y^n = x^m$

$$\frac{d}{dx} y^n = \frac{d}{dx} x^m$$

$$\frac{d}{dy} y^n \cdot \frac{dy}{dx} = m x^{m-1}$$

$$\frac{dy}{dx} = m x^{m-1} / n y^{n-1}$$

$$= \frac{m}{n} \frac{x^{m-1}}{(x^{m/n})^{n-1}} = \frac{m}{n} x^{\frac{m}{n}}$$

7. 高阶导数 $f^{(n)}(x) = \frac{d^ny}{dx^n} = \left(\frac{d}{dx}\right)^{(n)} y$

莱布尼茨: $(uv)^{(n)} = u^{(n)}v^{(0)} + C_n^1 u^{(n-1)}v^{(1)} + \dots + C_n^n u^{(0)}v^{(n)}$
 $= \sum_{k=0}^n C_n^k u^{(n-k)} v^{(k)}$

参数方程高阶导

$\begin{cases} x = u(t) \\ y = v(t) \end{cases} \cdot \frac{d^2y}{dx^2} = \frac{d \frac{dy}{dx}}{dx} = \frac{d \left(\frac{dy}{dt} / \frac{dx}{dt} \right)}{dx/dt} = \frac{v''(t)u'(t) - u''(t)v'(t)}{[u'(t)]^3}$

8. 极大值, 极小值 ($f'(x) = 0$, 局部峰值)

$f'(x) = 0$ 驻点:
 极大
 极小

9. 可导一定连续, 连续不一定可导

10. 中值定理: 闭区间连续, 开区间可导

Rolle: $f(a) = f(b)$, $\exists c \in (a, b)$, s.t. $f'(c) = 0$

Lagrange: $\exists c \in (a, b)$, s.t. $\frac{f(b) - f(a)}{b - a} = f'(c)$

Cauchy: $\begin{cases} y = f(t) \\ x = g(t) \end{cases}, t \in [a, b], \exists c \in (a, b)$, s.t. $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} = \frac{dy}{dx}$

e.g. $f(x)$ 在 $(0, +\infty)$ 可微, $g(x) = xf(\ln x)$, $0 < a, b$

证: $\exists c \in (a, b)$, s.t. $\frac{e^b f(b) - e^a f(a)}{e^b - e^a} = f(c) + f'(c)$

$g(e^b) = e^b f(b)$

$x_1 = e^b > e^a = x_0$, $g'(x) = f(\ln x) + f'(\ln x)$

由 Lagrange, $g(x)$ 在 $[x_0, x_1] \subset C$, (x_0, x_1) 可导, $\exists c \in (a, b)$ s.t. $\frac{e^b f(b) - e^a f(a)}{e^b - e^a} = f(c) + f'(c)$

0 洛必达: if $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, 在 a 附近 $\lim_{x \rightarrow a} \frac{f'}{g} = A$, then: $\lim_{x \rightarrow a} \frac{f}{g} = A$

由导: $f(a) = g(a) = 0$ (f, g 在 a 连续)

$\forall x \in U$, 不妨设 $x > a$, $\exists \varepsilon \in [a, x]$, s.t. $\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \Rightarrow \frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}$

$x \rightarrow a$, $\xi \rightarrow a$, $\lim_{x \rightarrow a} \frac{f}{g} = \lim_{\xi \rightarrow a} \frac{f'(\xi)}{g'(\xi)} = A$

11. Stolz 引理 (洛必达离散)

① $\frac{*}{\infty} \{b_n\} \nearrow \rightarrow \infty$, $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L$, s.t. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$

② $\frac{0}{0} \{b_n\} \searrow \rightarrow 0$, $\lim_{n \rightarrow \infty} a_n = 0$, $\dots$, s.t. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$

e.g. 证 $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}} = \frac{1}{p+1}$ ($p > 1$)

$a_n = 1^p + 2^p + \dots + n^p$, $b_n = n^{p+1}$, b_n 单调递增趋于无穷

$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \rightarrow \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} \rightarrow \frac{n^p}{(p+1)n^p} \rightarrow \frac{1}{p+1}$

Peano 余项

12. Taylor 公式: $f(x)$ 在 a 处 n 阶可导, s.t.

$f(x) = f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!} (x-a)^n + o[(x-a)^n]$

$\lim_{x \rightarrow a} \frac{o[(x-a)^n]}{(x-a)^n} = 0$
 高阶无穷小

(n+1) 阶可导.

$$f(x) = \dots + \frac{f^{(n+1)}(a)}{(n+1)!} (x-a)^{n+1} \rightarrow \text{Lagrange 余项}$$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k!} = e - 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + o(x^3)$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) \Rightarrow \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} = e^{-1} - 1$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + o(x^3)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^3)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)$$

$$\arcsin x = x + \frac{1}{2} \times \frac{x^3}{3} + \frac{1 \times 3}{2 \times 4} \times \frac{x^5}{5} + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \times \frac{x^7}{7} + o(x^7)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3)$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o(x^5)$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$

等价无穷小: -1 阶 Taylor 近似

求极限:

$$\lim_{x \rightarrow 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^6} = \lim_{x \rightarrow 0} \frac{\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^6)\right) - \left[1 - \frac{x^2}{2} + \frac{1}{2} \left(-\frac{x^2}{2}\right)^2 + o(x^6)\right]}{x^6}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{24}x^4 - \frac{x^4}{8} + o(x^6)}{x^6} = -\frac{1}{6}$$

极值: ① 左导/右导异号 / ② $f'(x_0) = 0$, $f''(x_0) > 0 \rightarrow$ 极小 $\cup$ $f\left(\frac{a+b}{2}\right) < \frac{1}{2}[f(a)+f(b)]$
 $f''(x_0) < 0 \rightarrow$ 极大 $\cap$ $f\left(\frac{a+b}{2}\right) > \frac{1}{2}[f(a)+f(b)]$

找最值: 极值 + 边界

$$\text{证 } \frac{x}{1+x} < \ln(1+x) < x$$

$$\leq f(x) = \ln(1+x) - x, f(0) = 0, f'(x) = \frac{1}{1+x} - 1$$

$$\forall x > 0, f'(x) < 0, f(x) \text{ 在 } (0, +\infty) \searrow, f(x) < f(0) = 0 \Rightarrow \ln(1+x) < x$$

$$\leq g(x) = \ln(1+x) - \frac{x}{1+x}, g(0) = 0, g'(x) > 0 (x > 0) \text{ 同理}$$

曲率: 有 $y = f(x)$, 曲率 $K = \frac{|f''(x)|}{[1 + f'^2(x)]^{3/2}}$

曲率半径 $r = \frac{1}{K}$ $r \downarrow$, 曲率 $\nearrow$

ej) $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-a|}$ ($a > 0$) 求最大值

三角函数变换

利用 $g(x) = \frac{1}{x}$ 绘图。

2. $y = \arctan x$, 证 $(1+x^2)f^{(n+2)} + 2(n+1)f^{(n+1)} + n(n+1)f^{(n)} = 0$

已知 $y' = \frac{1}{1+x^2} \Rightarrow (1+x^2)y' = 1$

$[(1+x^2)y']^{(n+1)} = 0 \Rightarrow$ 莱布尼茨展开, 证毕

3. $y_1 = 1 + e^{-x}$, $y_2 = (1+y)(1-y)$

$y_1' = y(1-y)$, $y_2' = 1-y$

4. $f(x) = \begin{cases} e^{-\frac{1}{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, 求 $f^{(n)}(0) = 0$

5. $f(x)$ 在 $(0, +\infty)$ 可微, if $\lim_{x \rightarrow \infty} f'(x) + f(x) = 0$, then: $\lim_{x \rightarrow \infty} f(x) = 0$

$x = \ln u$, $e^x = u \nearrow +\infty$

$\lim_{u \rightarrow +\infty} f'(\ln u) + f(\ln u) = 0 \rightarrow g(x) = xf(\ln x)$, $g'(u) \rightarrow 0$

$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} f(\ln x) = \lim_{x \rightarrow +\infty} \frac{g(x)}{x} \rightarrow g'(x) = 0$

6. $f(x)$ - 阶可微, $f'(x)$ 连续. $f(x) = 0$ 只有 l 个互不相等的根 $x_1 < x_2 < \dots < x_l$

(1) 利用 Taylor 证明, $\forall \epsilon > 0$, $\exists 0 < \epsilon_1 < \epsilon_2$, $0 < \epsilon_2 < \epsilon$, s.t. $\operatorname{sgn} f'(x_i + \epsilon_1) = -\operatorname{sgn} f'(x_{i+1} - \epsilon_2) \neq 0$ sgn : 取符号

$f(x) = f(x_0) + f'(x_0)(x-x_0) + o(x-x_0)^2$

令 $x_0 = x_i + \epsilon_1$, $f(x) =$

$f(x_i) = 0 \rightarrow 0 = f(x_i) = f(x_i + \epsilon_1) + f'(x_i + \epsilon_1)(-\epsilon_1) + o(\epsilon_1^2)$ $\nearrow$ 不影响正负

令 $x_0 = x_{i+1} - \epsilon_2$, 同理, $0 = f(x_{i+1}) = f(x_{i+1} - \epsilon_2) + f'(x_{i+1} - \epsilon_2)\epsilon_2 + o(\epsilon_2^2)$

易得 $f(x_i + \epsilon_1)$ 与 $f(x_{i+1} - \epsilon_2)$ 同号, 在其为正. 即 $\operatorname{sgn} f(x_i + \epsilon_1) = \operatorname{sgn} f(x_{i+1} - \epsilon_2) = 1$

$\therefore$ 证毕

(2) 证 $f'(x) = 0$ 至少有 $m \geq l-1$ 个根

(3) $g(x) = e^x f(x)$, $g'(x) = 0$ 有 j 个根, $f(x) + f'(x) = 0$ 有 n 个根. 证 $n = j$

$g'(x) = e^x f(x) + e^x f'(x) = e^x (f(x) + f'(x))$

$\forall x$, s.t. $f(x) + f'(x) = 0$, $\Rightarrow e^x (f(x) + f'(x)) = 0 \Rightarrow x$ 是 $g'(x)$ 的根

$\dots$ s.t. $g'(x) = 0 \quad \because e^x \neq 0 \quad \therefore \dots$

(4)

7. $y = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n$, 证: $(x^2-1)y'' + 2xy' - n(n+1)y = 0$

令 $u = (x^2-1)^n$

有 $u^{(n)} = 2^n \cdot n! \cdot y$, $u^{(n+1)} = 2^n \cdot n! \cdot y'$, $u^{(n+2)} = 2^n \cdot n! \cdot y'' \dots\dots (*)$

$u' = n(x^2-1)^{n-1} \cdot 2x$

$\Rightarrow (x^2-1)u' = 2nxu$, 2边用莱布尼茨展开, 代入*, 口

8.

16. *** (构造特定函数并使用 Lagrange 中值定理) 函数 $f(x), g(x), h(x)$ 在一个包含 $a, b (a < b)$ 的区间内可导, 证明, 在 (a, b) 区间内至少存在一个 c , 使得以下等式成立:

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0 \quad (1)$$

不定积分

不定积分

$$\begin{aligned} ① \int k dx &= kx + C \quad (k \text{ 是常数}), \\ ② \int x^\mu dx &= \frac{x^{\mu+1}}{\mu+1} + C \quad (\mu \neq -1), \\ ③ \int \frac{dx}{x} &= \ln|x| + C, \\ ④ \int \frac{dx}{1+x^2} &= \arctan x + C, \\ ⑤ \int \frac{dx}{\sqrt{1-x^2}} &= \arcsin x + C, \\ ⑥ \int \cos x dx &= \sin x + C, \\ ⑦ \int \sin x dx &= -\cos x + C, \\ ⑧ \int \frac{dx}{\cos^2 x} &= \int \sec^2 x dx = \tan x + C, \end{aligned}$$

$$\begin{aligned} ⑨ \int \frac{dx}{\sin^2 x} &= \int \csc^2 x dx = -\cot x + C, \\ ⑩ \int \sec x \tan x dx &= \sec x + C, \\ ⑪ \int \csc x \cot x dx &= -\csc x + C, \\ ⑫ \int e^x dx &= e^x + C, \\ ⑬ \int a^x dx &= \frac{a^x}{\ln a} + C. \end{aligned}$$

换元积分: 凑微分

$$\begin{aligned} \int \tan \theta d\theta &= \int \frac{\sin \theta}{\cos \theta} d\theta = \int \frac{d(-\cos \theta)}{\cos \theta} \xrightarrow{u=-\cos \theta} -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos \theta| + C \\ \int e^{kx} dx &= \frac{1}{k} \int e^{kx} dkx = \frac{1}{k} \cdot e^{kx} + C \\ \int \frac{dx}{x^2 - a^2} \quad (a \neq 0) &= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \frac{1}{2a} \ln|x-a| - \frac{1}{2a} \ln|x+a| + C \\ \int \sec x dx &= \int \frac{\cos x}{\cos^2 x} dx = \int \frac{\cos x}{1-\sin^2 x} dx = \int \frac{d\sin x}{1-\sin^2 x} \rightarrow \text{裂项} \\ &= \int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln|\sec x + \tan x| + C \end{aligned}$$

换元积分: 直接换元

$$\begin{aligned} \int \frac{du}{\sqrt{u^2 + a^2}} &= \int \frac{6x^5 dx}{x^2 + x^2} = \int \frac{6x^3}{x^2 + 1} dx = \int \frac{6(x^2 - \frac{x^4}{x^2+1})}{x^2 + 1} dx = \int (x^2 - \frac{x^4}{x^2+1}) dx = \int (\frac{1}{3}x^3 - \frac{1}{x^2+1}) dx = \frac{1}{12}x^4 - \ln|x^2+1| + C \\ \text{--- 三角换元} \quad \int \sqrt{a^2 - x^2} dx &= \int \sqrt{a^2 \cos^2 t} dt = a \int |\cos t| dt = \frac{a^2}{2} \int (1 + \cos 2t) dt = \frac{a^2}{2} (t + \frac{1}{2} \sin 2t) + C \xrightarrow{\text{换元}} = \frac{a^2}{2} (\arcsin \frac{x}{a} + \sin t \cos t) + C \\ \sin t = \frac{x}{a} \quad \frac{a}{1t} x &= \frac{a^2}{2} (\arcsin \frac{x}{a} + \frac{x}{a} \sqrt{1 - \frac{x^2}{a^2}}) + C \end{aligned}$$

$$\int \frac{dx}{x^2 - a^2} \xrightarrow{x = a \sec t} \frac{1}{a} \int \frac{a \sec^2 t dt}{\tan^2 t} = \int \sec^2 t dt = \ln|\sec t + \tan t| + C = \ln|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}| + C = \ln|x + \sqrt{x^2 - a^2}| + C$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} \xrightarrow{x = atant} \int \frac{a \sec^2 t dt}{(a^2(\tan^2 t + 1))^{3/2}} = \int \frac{a \sec^2 t dt}{a^3 \sec^3 t} = \frac{1}{a^2} \int \cos^2 t dt = \frac{1}{2a^2} \int (1 + \cos 2t) dt = \frac{1}{2a^2} (t + \frac{1}{2} \sin 2t) \xrightarrow{\text{带回}}$$

$$\int \frac{dx}{x \sqrt{x^2 - 1}} \xrightarrow{x = \sec t} \int \frac{\sec^2 t dt}{\sec^2 t - \tan^2 t} = \int \csc t dt = \ln|\csc t - \cot t| + C = \frac{\sqrt{x^2 - 1}}{x} + C$$

$$\xrightarrow{x = \frac{1}{u}} \int \frac{(-1)u^{-1} du}{\sqrt{1 - \frac{1}{u^2}}} = - \int \frac{u}{\sqrt{1 - u^2}} du \xrightarrow{(1-u^2)' = -2u} \sqrt{1 - u^2} + C = \sqrt{1 - \frac{1}{x^2}} + C$$

$$\int \frac{dx}{(x-2)\sqrt{4-x^2}} = \int \frac{dx}{\sqrt{1-(x-3)^2}} = \arcsin(x-3) + C$$

$$\sqrt{a^2 - x^2} \rightarrow x = a \sin t$$

$$\sqrt{a^2 + x^2} \rightarrow x = a \tan t$$

$$\sqrt{x^2 - a^2} \rightarrow x = a \sec t$$

分部积分:

$$\int u dv = uv - \int v du \quad \int (uv)' = \int (u'v + uv') \Rightarrow uv = \int u'v + \int uv'$$

$$\int \arctan x dx = \int \overset{v}{1} \cdot \overset{u}{\arctan x} dx = \overset{v}{x} \cdot \overset{u}{\arctan x} - \int \overset{v}{x} d\overset{u}{\arctan x} = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x + \frac{1}{2} \ln \frac{x^2+1}{x^2+1}$$

$$\int x e^{-x} dx = -e^{-x} x^1 - \int 2x e^{-x} dx = -x^2 e^{-x} + (-e^{-x} \cdot 2x) + \int 2e^{-x} dx$$

$p(x)$ 为多项式: ① $\int p(x) \sin^n x dx$ ② $\int p(x) e^{ax} dx$

e.g. $\int (x^4 + 2x + 1) e^{2x} dx = \frac{1}{2} (x^4 + 2x + 1) e^{2x} - \frac{1}{4} (4x^3 + 2) e^{2x} \dots$ e.g. $\int e^x \sin x dx = I$

$$\begin{array}{r} x^4 + 2x + 1 + e^{2x} \\ 4x^3 + 2 - \frac{1}{2} e^{2x} \\ 12x^2 + \frac{1}{4} e^{2x} \\ 24x + \frac{1}{8} e^{2x} \\ 24 + \frac{1}{16} e^{2x} \\ 0 + \frac{1}{32} e^{2x} \end{array}$$

$$\begin{aligned} I &= e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - (e^x \cos x + \int e^x \sin x dx) \\ &= e^x (\sin x - \cos x) - I \\ \Rightarrow I &= \frac{1}{2} e^x (\sin x - \cos x) \end{aligned}$$

导出递推式 $I_n = \int \frac{x^n}{\sqrt{1-x^2}} dx \quad (n \in \mathbb{Z}^+)$ $(\sqrt{1-x^2})' = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

$$\begin{aligned} &= \int x^{n-1} \frac{x}{\sqrt{1-x^2}} dx = x^{n-1} (-\sqrt{1-x^2}) + (n-1) \int x^{n-2} \sqrt{1-x^2} dx \\ &= -x^{n-1} \sqrt{1-x^2} + (n-1) \int x^{n-2} \frac{1-x^2}{\sqrt{1-x^2}} dx \\ &= -x^{n-1} \sqrt{1-x^2} + (n-1) (I_{n-2} - I_n) \end{aligned}$$

$$I_n = \int \frac{1}{(x^2+a^2)^n} dx \quad \int \frac{1}{(x^2+a^2)^n} dx = \frac{x}{(x^2+a^2)^n} - \int \frac{x \cdot (-n) \cdot (x^2+a^2)^{-n-1} \cdot 2x dx}{(x^2+a^2)^n} = \frac{x}{(x^2+a^2)^n} + 2n \int \frac{x^2+a^2}{(x^2+a^2)^{n+1}} dx = \frac{x}{(x^2+a^2)^n} + 2n I_n - 2a^2 n I_{n+1}$$

多项式有理数 拆

$$f(x) = \frac{p(x)}{(x-a)^n (x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}, \quad f(x) = \frac{p(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

可利用待定系数法

$$f(x) = \frac{p(x)}{(x-a)(x^2+bx+c)^n} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c} + \frac{Ex+D}{(x^2+bx+c)^2} + \dots$$

e.g. $\int \frac{x+1}{x^3+5x+6} dx = \frac{1}{2} \int \frac{(2x-5)dx}{x^3+5x+6} + \frac{7}{2} \int \frac{1}{x^3+5x+6} dx = \frac{1}{2} \ln |x^3+5x+6| + \frac{7}{2} \int \frac{1}{(x-3)(x^2-\frac{1}{2}x-2)} dx$

$$\int \frac{4x^4+2x^3+10x^2+3x+9}{(x+1)(x^2+2)^2} dx = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}$$

万能公式换元

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2t}{1+t^2}, \quad \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1-t^2}{1+t^2} \quad \tan \frac{x}{2} = t, \quad dx = \frac{2}{1+t^2} dt$$

e.g. $\int \frac{dx}{2+\sin x} \xrightarrow{u=\tan \frac{x}{2}} \int \frac{1}{2+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{du}{1+u+u^2} = \int \frac{du}{(u+\frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \arctan \frac{2u+1}{\sqrt{3}} + C =$

积化和差

△ e.g. $\int \sin \alpha x \sin \beta x dx$

无理根式

换元整个根式

$$e.g. \int \frac{1}{x \sqrt{x^2-2}} dx \xrightarrow{t=\frac{\sqrt{x^2-2}}{x}} \int \frac{1}{(1+t^2)\sqrt{2+t^2-x^2}} dx = \int \frac{1}{(1+t^2)\sqrt{2-t^2}} dt =$$

无理根式的三角换元. 欧拉变换

$$\begin{aligned} & \int \frac{1}{x\sqrt{x^2-2x-3}} dx \\ &= \int \frac{1}{x\sqrt{(x-1)^2-4}} dx \quad x-1=u \\ &= \int \frac{1}{(u+1)\sqrt{u^2-4}} du \quad u=2\sec\theta \\ &= \int \frac{2\sec\theta \tan\theta d\theta}{(2\sec\theta+1)2\tan\theta} = \int \frac{\sec\theta d\theta}{2\sec\theta+1} = \int \frac{d\theta}{2+\cos\theta} \quad \text{万能公式} \\ &= \int \frac{\frac{2}{1+\cos\theta}}{2+\frac{1+\cos\theta}{2}} d\theta = \int \frac{2d\theta}{t^2+3} = \frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C \end{aligned}$$

用欧拉变换

$$\begin{aligned} \sqrt{x^2-2x-3} &= x-t \Rightarrow t^2+3 = x(2t-2) \Rightarrow dx = \frac{t^2-2t-3}{2(t-1)^2} \cdot \sqrt{x^2-2x-3} = x-t = \frac{t^2+3}{2t-2} - t = \frac{-(t^2-2t-3)}{2(t-1)} \\ \text{LHS} &= \int \frac{2t-2}{t^2+3} \cdot \frac{2(t-1)}{-t^2+2t-3} \cdot \frac{t^2-2t-3}{2(t-1)^2} dt = -\int \frac{2}{t^2+3} dt = -\frac{2}{\sqrt{3}} \arctan \frac{t}{\sqrt{3}} + C \quad \text{同A} \end{aligned}$$

$$\sqrt{ax^2+bx+c} = \sqrt{ax} \pm t \quad (a>0); \sqrt{ax^2+bx+c} = xt \pm \sqrt{c} \quad (c>0)$$

定积分 $\int_a^b f(x) dx = F(b) - F(a), \int f(x) dx = F(x) + C$

$$\int_a^b f(x) dx = \int_a^b f(t) dt$$

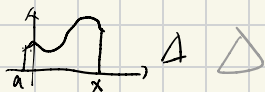
定积分求极限 $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \Delta x_i$

$$f(x) = \frac{1}{1+x}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \right) &\rightarrow \frac{1}{n} \left(\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots + \frac{1}{1+\frac{n}{n}} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+\frac{i}{n}} = \int_0^1 \frac{1}{1+x} dx = \ln 2 \\ \lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2} \right) &\rightarrow \frac{1}{n^2} \left(\frac{1}{1+(\frac{1}{n})^2} + \frac{1}{1+(\frac{2}{n})^2} + \dots + \frac{1}{1+(\frac{n}{n})^2} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \frac{1}{1+(\frac{i}{n})^2} = \int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4} \end{aligned}$$

变上限积分

$$F(x) = \int_a^x f(t) dt, F'(x) = f(x); \quad F(x) = \int_x^b f(t) dt, F'(x) = -f(x); \quad F(x) = \int_a^{\phi(x)} f(t) dt, F'(x) = f(\phi(x)) \phi'(x) - f(a(x)) a'(x)$$



函数翻折 $\int_0^a f(x) dx = \int_0^a f(a-x) dx, \int_0^a f(x) dx = \int_0^a [f(x) + f(a-x)] dx$

$$\begin{aligned} & \int_0^{\frac{\pi}{2}} \ln(1+\tan x) dx \\ &= \int_0^{\frac{\pi}{2}} \ln(1+\tan(\frac{\pi}{2}-x)) dx = \int_0^{\frac{\pi}{2}} \ln\left(1+\frac{\tan\frac{\pi}{2}-\tan x}{1+\tan\frac{\pi}{2}\tan x}\right) dx = \int_0^{\frac{\pi}{2}} \ln \frac{2}{1+\tan x} dx = \frac{\pi}{2} \ln 2 - \int_0^{\frac{\pi}{2}} \ln(1+\tan x) dx \end{aligned}$$

奇偶函数 $\int_{-a}^a f(x) dx = \begin{cases} 0, & \text{奇} \\ 2\int_0^a f(x) dx, & \text{偶} \end{cases}$

证明 Δ 周期函数 $\int_a^{a+T} f(x) dx = \int_a^T f(x) dx$

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx, \int_0^{\pi} f(\sin x) dx = 2 \int_0^{\frac{\pi}{2}} f(\sin x) dx, \int_0^{\pi} f(\cos x) dx = 2 \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx = \pi \int_0^{\frac{\pi}{2}} f(\sin x) dx$$

正交函数系

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = \int_{-\pi}^{\pi} \cos mx \sin nx dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases} \quad \text{积化和差}$$

点火公式 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = f(n) = \begin{cases} \frac{(2m)!!}{(2m+1)!!} & n=2m+1 \\ \frac{(2m-1)!!}{2m!!} \cdot \frac{\pi}{2} & n=2m \end{cases}$

证明: $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x dx = -\cos x \sin^{\frac{n-1}{2}} x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot (n-1) \sin^{n-2} x \cdot \cos x dx$
 $= 0 + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cdot (1 - \sin^2 x) dx = (n-1) I_{n-2} - (n-1) I_n \Rightarrow I_n = \frac{n-1}{n} I_{n-2}, I_0 = \frac{\pi}{2}, I_1 = 1$

Wallis 公式 $\frac{\pi}{2} = \lim_{m \rightarrow \infty} \frac{(2m)!!}{(2m-1)!!} \cdot \frac{1}{2m+1}$

证明: 已知

$$\int_0^{\frac{\pi}{2}} \sin^{2m} x dx < \int_0^{\frac{\pi}{2}} \sin^{2m-2} x dx < \int_0^{\frac{\pi}{2}} \sin^{2m-4} x dx \quad (\forall m)$$

点火!! $\frac{(2m)!!}{(2m-1)!!} \leq \frac{(2m-1)!!}{(2m-2)!!} \cdot \frac{\pi}{2} \leq \frac{(2m-1)!!}{(2m-3)!!}$

$$A_m = \frac{1}{2m+1} \left[\frac{(2m)!!}{(2m-1)!!} \right]^2 \leq \frac{\pi}{2} \leq \left(\frac{(2m-1)!!}{(2m-2)!!} \right)^2 \cdot \frac{1}{2m} = B_m$$

$$B_m + A_m = \frac{(2m)!!}{(2m-1)!!} \cdot \frac{1}{(2m)(2m-1)} = A_m \cdot \frac{1}{2m} \leq \frac{\pi}{2} \cdot \frac{1}{2m} \rightarrow 0 \quad (m \rightarrow \infty)$$

$$\lim_{m \rightarrow \infty} (B_m - A_m) = 0 \Rightarrow 0 \leq \frac{\pi}{2} - A_m \leq B_m - A_m \quad (\text{夹逼})$$

平面面积

极坐标 $S = \frac{1}{2} \int_a^b \rho^2(\theta) d\theta$ $S = \frac{1}{2} \rho(\theta) \cdot dL$, $\theta = \frac{360^\circ}{2\pi} \frac{1}{\rho(\theta)} d\theta$, $d\theta = \frac{dL}{\rho(\theta)}$

曲线长度 $s = \int_a^b \sqrt{1+y'^2} dx$ $S = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

参数方程: $s = \int_a^b \sqrt{u'(t)^2 + v'(t)^2} dt$ $S = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

极坐标: $S = \int_a^b \sqrt{\rho^2 + \rho'^2} d\theta$

$$\begin{cases} x(\theta) = \rho(\theta) \cos \theta \\ y(\theta) = \rho(\theta) \sin \theta \end{cases} \Rightarrow S = \int_a^b \sqrt{x(\theta)^2 + y(\theta)^2} d\theta$$

旋转体体积

$y = x^k (1 \leq x \leq 5)$ 绕 x 轴围成的体积. $V = \int_1^5 \pi f^2(x) dx$ (切片)

绕 y 轴.

$$V = \int_a^b \pi (f(y))^2 dy$$

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{c}{a}} = 1$$

$z=h \Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = 1 - \sqrt{\frac{c}{a}} = k$ 即 $z=h$ 面上曲线的方程

$ky=0 \Rightarrow x=ak^2 \Rightarrow S = \int_0^{ak^2} f(x) dx$

$\sqrt{\frac{a}{b}} = k - \sqrt{\frac{c}{a}}$, $y = bk^2 - 2kb\sqrt{\frac{a}{c}} + \frac{b}{a}x$, 即 $f(x)$. 则 $S = \int_0^{ak^2} (bk^2 - 2kb\sqrt{\frac{a}{c}} + \frac{a}{b}x) dx = \frac{1}{b}abk^3 = \frac{1}{b}ab(1-\sqrt{\frac{c}{a}})^3$

$V = \int_0^c S dh = \int_0^c \frac{1}{b}ab(1-\sqrt{\frac{c}{a}})^3 dh =$

旋转体侧面积

$$S = 2\pi \int_a^b f(x) \sqrt{1+f'(x)^2} dx = 2\pi \int_a^b y(t) \sqrt{x'(t)^2 + y'(t)^2} dt \quad (x \text{ 轴})$$

证明①: 360微分 垂直 x 轴切片

$$S = \int_a^b 2\pi f(x) ds = \int_a^b 2\pi f(x) \sqrt{1+f'(x)^2} dx$$

证明②: 圆台

$\begin{matrix} f_1(x) = f_1 \\ f_2(x) = f_2 \end{matrix}$ $S(\text{圆台}) = \frac{(2\pi r_1 + 2\pi r_2) \cdot \Delta l}{2} = \pi(f(x) + f(x+\Delta x)) \Delta s$

$$\Delta S = S(\text{圆台}) = \pi(f(x) + f(x+\Delta y)) \sqrt{\Delta x^2 + \Delta y^2} = \pi(2f(x) + \Delta y) \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x \approx 2\pi f(x) \sqrt{1+f'(x)^2} \Delta x$$

$$S = \int_a^b ds = \int_a^b \Delta S$$

定积分不等式

$$|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx, \quad f(x) \leq g(x) \rightarrow \int_a^b f(x) dx \leq \int_a^b g(x) dx, \quad c \geq b, f(x) \geq 0 \rightarrow \int_a^b f(x) dx \leq \int_a^c f(x) dx$$

$$m \leq f(x) \leq M \rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

柯西不等式 $方和积 \geq 积和方$

$$(\int_a^b f(x)g(x)dx)^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$

证明: $\varphi(t) = t^2 \int_a^b f^2(x) dx + 2t \int_a^b f(x)g(x) dx + \int_a^b g^2(x) dx$

$$= \int_a^b [f(x) + g(x)]^2 dx \geq 0 \Rightarrow \Delta \geq 0$$

$$\Delta = (\int_a^b f \cdot g \, dx)^2 - 4 \int_a^b f^2 \, dx \cdot \int_a^b g^2 \, dx \leq 0 \Rightarrow \int_a^b f(x)g(x) \, dx \leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \quad \square$$

积分中值定理

$f(x)$ 在 $[a, b]$ 连续, $\exists \xi \in (a, b)$, s.t. $\int_a^b f(t) dt = f(\xi)(b-a)$

$f(x)$ 在 $[a, b]$ 连续, $g(x)$ 在 $[a, b]$ 不变号, $\exists \xi \in (a, b)$, s.t. $\int_a^b f(x)g(x)dx = \underbrace{f(\xi)}_{\substack{\uparrow \\ \text{中值}}}\int_a^b g(x)dx$

反常积分

无限积分: $J = \int_a^{+\infty} f(x) dx = \lim_{u \rightarrow +\infty} \int_a^u f(x)$, $J = \int_{-\infty}^b f(x) dx = \lim_{u \rightarrow -\infty} \int_u^b f(x) dx$

$$J = \int_{-\infty}^{+\infty} f(x) dx = \lim_{u \rightarrow -\infty} \int_u^a f(x) dx + \lim_{u \rightarrow +\infty} \int_a^u f(x) dx$$

$$\int_1^{+\infty} \frac{1}{x^p} dx = \lim_{u \rightarrow +\infty} \int_1^u \frac{1}{x^p} dx$$

对于 $\int_1^u \frac{1}{x^p} dx = \begin{cases} \ln|u|, & p=1 \\ \frac{1}{1-p}(u^{1-p}-1), & p \neq 1 \end{cases}$

$$\int_1^{+\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{p-1}, & p > 1 \text{ 收敛} \\ +\infty, & p \leq 1 \text{ 发散} \end{cases}$$

e.g. $f(1)=1$, $f'(x) = \frac{1}{x^2 + f^2(x)}$, i.e. $\forall x > 1$, s.t. $f(x) < 1 + \frac{1}{x}$

即证 $f(x) - f(1) < \frac{\pi}{4} \Leftrightarrow \int_1^x f'(x) dx < \frac{\pi}{4} \Leftrightarrow \int_1^x \frac{1}{x^2 + f^2(x)} dx < \frac{\pi}{4}$

$$\therefore f'(x) > 0 \quad \therefore x > 1, f(x) > f(1) = 1$$

有: $\int_1^{+\infty} \frac{dt}{t^2 + f^2(t)} < \int_1^{+\infty} \frac{1}{t^2 + 1} dt = \frac{\pi}{4}$

$$\frac{1}{t^2 + f^2(t)} > 0, \text{ 有 } \int_1^x \frac{dt}{t^2 + f^2(t)} < \int_1^{+\infty} \frac{dt}{t^2 + f^2(t)} < \frac{\pi}{4} \quad \square$$

瑕积分

$$\int_a^b f(x) dx = \lim_{u \rightarrow b^-} \int_a^u f(x) dx / \lim_{u \rightarrow a^+} \int_u^b f(x) dx \quad \text{1个奇点}$$

$$\int_a^b f(x) dx = \lim_{u \rightarrow a^+} \int_u^c f(x) dx + \lim_{u \rightarrow b^-} \int_c^u f(x) dx \quad \text{2个奇点}$$

$$= \lim_{a \rightarrow c^-} \int_a^c f(x) dx + \lim_{a \rightarrow c^+} \int_c^b f(x) dx \quad \text{中间奇点}$$

$$\int_0^1 \frac{1}{x^q} dx \quad (q > 0)$$

$$\int_0^1 \frac{1}{x^q} dx = \begin{cases} -\ln(u), & q=0 \\ \frac{1}{1-q}(1-u^{1-q}), & q \neq 1 \end{cases}, \quad \int_0^\infty \frac{1}{x^q} dx = \begin{cases} \frac{1}{1-q}, & 0 < q < 1 \\ \infty, & q \geq 1 \end{cases}$$

无限积分收敛

$$\int_a^{+\infty} f(x) dx \text{ 收敛} \Leftrightarrow \forall \varepsilon > 0, \exists G \geq a, \text{ s.t. } \forall u_1, u_2 > G, \text{ 有 } \left| \int_{u_1}^{u_2} f(x) dx \right| < \varepsilon$$

$b > a$, 则 $\int_a^{+\infty} f(x) dx, \int_b^{+\infty} f(x) dx$ 同敛散

$\int_a^{+\infty} f_1(x) dx, \int_a^{+\infty} f_2(x) dx$ 收敛, 则 $\int_a^{+\infty} f_1(x) + f_2(x) dx$ 收敛

$$\int_1^{+\infty} f(x) dx \text{ 收敛, 则 } \lim_{x \rightarrow +\infty} f(x) = 0$$

$$\int_a^{+\infty} f(x) dx \text{ 收敛} \longrightarrow \int_a^{+\infty} f(x) dx \text{ 收敛} \quad \left\{ \begin{array}{l} \int_a^{+\infty} f(x) dx \text{ 收敛}, \int_a^{+\infty} |f(x)| dx \text{ 收敛, 绝对收敛} \\ - \quad - \quad - \quad - \quad - \quad \text{发散, 条件收敛} \end{array} \right.$$

非负元函数收敛

$$f \leq g, \int_a^{+\infty} g dx \text{ 收} \rightarrow \int_a^{+\infty} f dx \text{ 收}; \int_a^{+\infty} f dx \text{ 发} \rightarrow \int_a^{+\infty} g dx \text{ 发}$$

$$\int_0^{+\infty} \frac{\sin x}{1+x^2} dx, \frac{\sin x}{1+x^2} \leq \left| \frac{\sin x}{1+x^2} \right| \leq \frac{1}{1+x^2}, \int_0^{+\infty} \frac{1}{1+x^2} dx = \lim_{u \rightarrow +\infty} \arcsin x \Big|_0^u = \frac{\pi}{2} \text{ 收}$$

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = C, \text{ ① } 0 < C < +\infty, f, g \text{ 同敛散} \text{ ② } C = 0, \int g \text{ 收} \rightarrow \int f \text{ 收} \text{ ③ } C = +\infty, \int f \text{ 收} \rightarrow \int g \text{ 收}$$

$$\text{常用轴R} \int_1^{+\infty} \frac{1}{x^p} dx$$

瑕积分收敛

$$a \text{ 为奇点} \text{ ① } \int_a^b f(x) dx \text{ 收敛} \iff \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall u, v \in (a, a+\delta), \left| \int_u^v f(x) dx \right| < \varepsilon$$

$$\text{② } c \in (a, b), \int_a^c f(x) dx, \int_c^b f(x) dx \text{ 同敛散}$$

$$\text{③ } \int_a^b f, \int_a^b g \text{ 收} \rightarrow \int_a^b f+g \text{ 收}$$

绝对收敛, 条件收敛

比较原则 (比值原则) ✓

$$\text{标尺 } \int_a^b \frac{1}{(x-a)^p} dx \text{ 收敛}$$

$$0 \leq f \leq \frac{1}{(x-a)^p}, 0 < p < 1, \text{ 收}; f \geq \frac{1}{(x-a)^p}, p \geq 1, \text{ 散}$$

$$\lim_{x \rightarrow a^+} (x-a)^p f(x) = \lambda \text{ ① } 0 < p < 1, 0 \leq \lambda < +\infty, \text{ 收} \text{ ② } p \geq 1, 0 < \lambda < +\infty, \text{ 散}$$

欧拉积分

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx \quad (\alpha > 0) \text{ 含参变量积分 } \Gamma(1) = 1$$

$$x = t^{\frac{1}{n}} = 2 \int_0^{+\infty} t^{\frac{1}{n}-1} e^{-t^{\frac{1}{n}}} dt \quad \Gamma\left(\frac{1}{n}\right) = 2 \int_0^{+\infty} e^{-t^{\frac{1}{n}}} dt = \sqrt{n} \text{ 需要二重积分}$$

$$x = \ln \frac{1}{t} = \int_0^{+\infty} (\ln \frac{1}{t})^{\alpha-1} dt$$

$$\text{有: } \Gamma^{(n)}(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} (\ln x)^n dx \text{ 对 } \alpha \text{ 求 } n \text{ 阶导}$$

$$LHS = \frac{d^n}{d\alpha^n} \Gamma(\alpha) = \frac{d^n}{d\alpha^n} \int_0^{+\infty} x^{\alpha-1} e^{-x} dx = \int_0^{+\infty} \frac{d^n}{d\alpha^n} (x^{\alpha-1} e^{-x}) dx \text{ 积分, 微分互换证明略 } (\alpha^x)^{(n)} = \alpha^x (\ln \alpha)^n$$

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha) \quad \Gamma(n) = (n-1)!, n \in \mathbb{Z}^+$$

$$LHS = \int_0^{+\infty} x^{\alpha} e^{-x} dx = -x^{\alpha} e^{-x} \Big|_0^{+\infty} + \alpha \int_0^{+\infty} x^{\alpha-1} e^{-x} dx = \alpha \Gamma(\alpha)$$

$$\Gamma(\alpha) \Gamma(1-\alpha) = \frac{\pi}{\sin \alpha \pi} \quad (0 < \alpha < 1) \text{ 先不证}$$

$$\Gamma(2\alpha) = \frac{2^{2\alpha-1}}{\sqrt{\pi}} \Gamma(\alpha) \Gamma(\alpha + \frac{1}{2})$$

Beta函数

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx \quad (p, q > 0) \quad B(p, q) = B(q, p)$$

$$x = \cos^2 \theta = 2 \int_0^{\frac{\pi}{2}} \cos^{2p-1} \theta \sin^{2q-1} \theta d\theta$$

$$x = \frac{u}{1+u} = \int_0^{+\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du$$

$$= \int_0^{+\infty} \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du$$

$$\hookrightarrow \int_0^{+\infty} \frac{u^{p-1}}{(1+u)^{p+q}} du + \int_0^{+\infty} \frac{u^{q-1}}{(1+u)^{p+q}} du = I_1 + I_2, I_2 \xrightarrow{u = \frac{1}{t}} \int_1^{+\infty} \frac{(\frac{1}{t})^{q-1}}{(\frac{1}{t}+1)^{p+q}} d\frac{1}{t} = \int_1^{+\infty} \frac{t^{q-1}}{(1+t)^{p+q}} dt = \int_0^1 \frac{t^{q-1}}{(1+t)^{p+q}} dt = \int_0^1 \frac{t^{q-1}}{(1+t)^{p+q}} dt$$

$$LHS = I_1 + I_2 = \int_0^{+\infty} \frac{u^{p-1} + u^{q-1}}{(1+u)^{p+q}} du$$

$$\text{有: } B(p, q) = B(q, p)$$

$$B(p, q) = \frac{\Gamma(p) \Gamma(q)}{\Gamma(p+q)} \text{ 先不证}$$

$$\downarrow$$

$$B(p, 1-p) = \frac{\pi}{\sin p \pi} \quad (0 < p < 1)$$

$$B(p, q) = \frac{p-1}{p+q-1} B(p, q-1) = \frac{p-1}{p+q-1} B(p-1, q)$$

$$\begin{aligned} \text{LHS} &= \frac{p-1}{p} \int_0^1 (1-x)^{q-1} \left|_0^1 + \frac{q-1}{p} \int_0^1 x^p (1-x)^{q-2} dx \right. \\ &= \frac{q-1}{p} \int_0^1 [x^{p-1} - x^{p-1} (1-x)] (1-x)^{q-2} dx = \frac{q-1}{p} \left[\int_0^1 x^{p-1} (1-x)^{q-2} dx - \int_0^1 x^{p-1} (1-x)^{q-1} dx \right] \\ &= \frac{q-1}{p} [B(p, q-1) - B(p, q)] \end{aligned}$$

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

$$B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi \frac{\frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(1)}}{1} = \frac{(\sqrt{\pi})^2}{1} = \pi$$

$$\text{e.g. } \int_0^1 x^{\frac{p}{m}} (1-x)^{\frac{q}{m}} dx \quad x^{\frac{p}{m}} = u, \quad dx = \frac{1}{m} u^{\frac{m}{m}-1} du \quad \int_0^1 \frac{dx}{\sqrt{1-x}} \cdot \int_0^1 \frac{x^2 dx}{1-x^2} = I_1 \cdot I_2 = \frac{\pi}{4}$$

$$= \int_0^1 u^{\frac{p}{m}} (1-u)^{\frac{q}{m}} \cdot \frac{1}{m} \cdot u^{\frac{m}{m}-1} du$$

$$I_1 = \int_0^1 x^{1-1} (1-x^2)^{\frac{1}{2}-1} dx = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{m} \int_0^1 u^{\frac{p}{m}} (1-u)^{\frac{q}{m}} du = \frac{1}{m} B\left(\frac{p}{m}, \frac{q}{m}\right)$$

$$\int_0^1 \frac{x^{m-1}}{1+x^n} dx \quad (n > m > 0) \quad x^n = u, \quad dx = \frac{1}{n} u^{\frac{1}{n}-1} du$$

$$= \frac{1}{n} \int_0^1 \frac{u^{\frac{m}{n}-1}}{1+u} \cdot u^{\frac{1}{n}-1} du = \frac{1}{n} \int_0^1 \frac{u^{\frac{m}{n}-1}}{1+u} du = \frac{1}{n} B\left(\frac{m}{n}, 1-\frac{m}{n}\right) = \frac{1}{n} \cdot \frac{\pi}{\sin \frac{m\pi}{n}}$$

$$\int_0^1 \frac{dx}{(1-x^n)^{n-1}} \quad (n \geq 1)$$

$$= \frac{1}{n} \int_0^1 t^{\frac{1}{n}-1} (1-t)^{\frac{1}{n}-1} dt = \frac{1}{n} B\left(\frac{1}{n}, 1-\frac{1}{n}\right)$$

$$\text{证明} \int_0^{\infty} x^n e^{-x^2} dx = \frac{n-1}{2} \int_0^{\infty} x^{n-2} e^{-x^2} dx$$

欧拉积分法一次

$$\begin{aligned} \text{LHS} &= \frac{1}{2} \cdot 2 \int_0^{\infty} x^{2 \cdot \frac{n+1}{2}-1} \cdot e^{-x^2} dx = \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) = \frac{1}{2} \cdot \frac{n-1}{2} \cdot \Gamma\left(\frac{n-1}{2}\right) \\ &= \frac{n-1}{2} \int_0^{\infty} x^{n-2} e^{-x^2} dx \end{aligned}$$

靠怎么又是习题课

$$1. \int_{-1}^1 \frac{1}{\sqrt{1-x}} dx = \int_{-1}^0 \frac{1}{\sqrt{1-x}} dx + \int_0^1 \frac{1}{\sqrt{1-x}} dx = (-2\sqrt{1-x}) \Big|_{-1}^0 + (2\sqrt{1-x}) \Big|_0^1 = 4$$

$$\begin{aligned} 2. \int_0^{\sin \theta} \frac{\arctan(\arcsin x)}{\sqrt{1-x^2}} dx \quad \arcsin x = t, \quad x = \sin t, \quad dx = \cos t, \quad dt = \frac{1}{\sqrt{1-x^2}} dx \\ = \int_0^{\theta} \arctan t \cdot \cos t dt = t \arctan t \Big|_0^{\theta} - \int_0^{\theta} \frac{t}{1+t^2} dt = \theta \arctan \theta - \frac{1}{2} \ln |1+t^2| \Big|_0^{\theta} = \theta \arctan \theta - \frac{1}{2} \ln |1+\theta^2| \end{aligned}$$

$$3. \int_1^{\infty} \frac{2+\sin e^x}{x} dx \quad \text{不在} \wedge$$

$$4. f(x) = \exp(-x - e^{-x}), \quad -\infty < x < +\infty, \quad F(x) = \int_{-\infty}^x f(t) dt, \quad u = -e^{-t}$$

$$-t = \ln(-u), \quad dt = -\frac{1}{u} du$$

$$F(x) = \int_{-\infty}^x e^{-t} \cdot e^{-e^{-t}} dt = \int_{-\infty}^{-e^{-x}} (-u) \cdot e^u \cdot \left(-\frac{1}{u}\right) du = \int_{-\infty}^{-e^{-x}} e^u du = e^{-e^{-x}}$$

$$\text{求} \int_{-\infty}^{\infty} F(x) f(x) dx$$

$$= \int_{-\infty}^{\infty} e^{-x-2e^{-x}} dx \quad u = -e^{-x} \rightarrow \int_{-\infty}^0 (-u) \cdot e^{2u} \cdot \left(-\frac{1}{u}\right) du =$$

$$5. \int_0^{\frac{\pi}{2}} \cos 2\theta \ln \cos \theta d\theta$$

$$= \frac{1}{2} \sin 2\theta \cdot \ln \cos \theta \Big|_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{\cos \theta} (-\sin \theta) d\theta$$

$$\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{1}{2} \sin 2\theta \ln \cos \theta = \frac{1}{2} \lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\ln \cos \theta}{1/\sin \theta} \rightarrow \frac{1}{2} \frac{-\tan \theta}{\frac{-2 \cos \theta}{\sin^2 \theta}} \rightarrow \frac{\sin^2 \theta \sin 2\theta}{2 \cos 2\theta} \rightarrow \frac{0}{-2} = 0$$

$$= 0 + \frac{1}{2} \int_0^{\frac{\pi}{2}} 2 \sin^2 \theta d\theta = \frac{\pi}{4}$$

$$6. \begin{cases} x = a \cos^3 t \\ y = a \sin^3 t \end{cases}, \quad 0 \leq t \leq 2\pi, \quad \text{求面积、周长}$$

欧/Beta

$$S = 4S' = 4 \cdot \int_0^a y dx = 4 \cdot \int_{\frac{\pi}{2}}^0 b \sin^3 t \cdot a \cdot 3 \cos^2 t (-\sin t) dt = 12a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt = 12a^2 \cdot \frac{\pi}{32} = \frac{3a^2 \pi}{8}$$

$$C = 4C' = 4 \int_0^{\frac{\pi}{2}} \sqrt{(dx/dt)^2 + (dy/dt)^2} dt = 6a$$

$$\text{复数} \int_0^{\infty} x e^{-x} \sin(bx) dx \quad e^i = \cos + i \sin \quad \text{LHS} = \beta \leftarrow$$

$$A = \int_0^{\infty} x e^{-x} e^{ibx} dx = \int_0^{\infty} x e^{(-1+ib)x} dx = \int_0^{\infty} x e^{-x} (\cos bx + i \sin bx) dx = \alpha + \beta i$$

...

$$\begin{aligned}
 8. \int_0^{+\infty} e^{-yw} \cos(xw) dw \\
 &= \int_0^{+\infty} \cos(xw) \cdot \left(-\frac{1}{y}\right) e^{-yw} dw = -\frac{1}{y} e^{-yw} \cos(xw) \Big|_0^{+\infty} - \int_0^{+\infty} \frac{1}{y} e^{-yw} x \sin(xw) dw \\
 &= 0 - \left(-\frac{1}{y}\right) - \frac{x}{y} \int_0^{+\infty} e^{-yw} \sin(xw) dw \\
 &= \frac{1}{y} - \frac{x}{y} \left[-\frac{1}{y} e^{-yw} \sin(xw) \Big|_0^{+\infty} + \frac{x}{y} \int_0^{+\infty} e^{-yw} \cos(xw) dw \right] = \frac{1}{y} - \frac{x}{y} [0 + \frac{x}{y} \text{LHS}]
 \end{aligned}$$

$$9. \int_0^{+\infty} f(ax - \frac{b}{x}) dx = \frac{1}{2a} \int_{-\infty}^{+\infty} f(x) dx \quad a, b > 0, f(x) \text{ 为偶函数}$$

$$x = \frac{b}{at}, \quad dx = -\frac{b}{at^2} dt$$

$$I = \int_0^{+\infty} f(ax - \frac{b}{x}) dx = \int_{+\infty}^0 f(\frac{b}{t} - at) \cdot \frac{b}{at^2} dt$$

$$aI = \int_0^{+\infty} af(ax - \frac{b}{x}) dx = \int_0^{+\infty} f(at - \frac{b}{x}) \frac{b}{x} dt = \int_0^{+\infty} f(ax - \frac{b}{x}) \frac{b}{x} dx$$

$$2aI = \int_0^{+\infty} (at + \frac{b}{x}) f(ax - \frac{b}{x}) dx = \int_0^{+\infty} f(ax - \frac{b}{x}) d(ax - \frac{b}{x}) \quad 0 < x < +\infty, -\infty < ax - \frac{b}{x} < +\infty$$

$$I = \frac{1}{2a} \int_{-\infty}^{+\infty} f(x) dx \quad \square$$

$$10. I(m, n) = \int_0^1 x^m (\ln x)^n dx$$

$$\begin{aligned}
 &= \frac{1}{m+1} x^{m+1} (\ln x)^n \Big|_0^1 - \int_0^1 \frac{x^{m+1}}{m+1} \cdot \frac{n}{x} (\ln x)^{n-1} dx \quad \lim_{x \rightarrow 0^+} \frac{x^{m+1}}{m+1} (\ln x)^n \stackrel{x=\frac{1}{t}}{=} \lim_{t \rightarrow +\infty} \frac{1}{t^{m+1}} \frac{(-\ln t)^n}{t} = 0 \quad \text{多项式} \gg \text{对数} \\
 &= 0 - \frac{n}{m+1} \int_0^1 x^m (\ln x)^{n-1} dx = -\frac{n}{m+1} I(m, n-1) \quad \text{递推}
 \end{aligned}$$

$$I(m, n) = (-1)^n \cdot \frac{n!}{(m+1)^n} I(m, 0) = (-1)^n \frac{n!}{(m+1)^n} \int_0^1 x^m dx = (-1)^n \frac{n!}{(m+1)^{n+1}} \quad \square$$

$$11. I(m) = \int_0^1 x^m \arccos x dx$$

$$\begin{aligned}
 &= \frac{x^{m+1}}{m+1} \arccos x \Big|_0^1 + \int_0^1 \frac{1}{m+1} \cdot x^{m+1} \cdot \frac{1}{\sqrt{1-x^2}} dx \\
 &= 0 + \frac{1}{m+1} \int_0^1 x^{m+1} (1-x^2)^{-\frac{1}{2}} dx \xrightarrow{x=\sin \theta} \frac{1}{m+1} \int_0^{\frac{\pi}{2}} \sin^{m+2} \theta d\theta \stackrel{\text{递推}}{=} \left\{
 \end{aligned}$$

无穷级数

级数运算。

级数收敛，改变有限个值、改变运算顺序不影响收敛值。

$$\sum_{n=1}^{\infty} a_n = S \rightarrow \lim_{n \rightarrow \infty} a_n = 0, \lim_{n \rightarrow \infty} S_n \rightarrow S$$

$$|S_{n+p} - S_n| < \varepsilon$$

$$\text{Cauchy 收敛准则} \quad \sum_{n=1}^{\infty} a_n = S \iff \forall \varepsilon > 0, \exists N > 0, \text{ if } n > N, \text{ s.t. } \forall p \in \mathbb{N}^+, |a_{n+1} + \dots + a_{n+p}| < \varepsilon$$

S_N 部分和收敛 $\leftrightarrow$ 级数收敛

常用标度

$$\textcircled{1} \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad |q| < 1,$$

无穷等比数列求和

$$\textcircled{2} \sum_{n=1}^{\infty} \frac{1}{n^p}: \quad p > 1 \text{ 收敛}$$

$$\text{散} \begin{cases} p=1 & \text{调和级数 } \sum \frac{1}{n} \quad S_N = 1 + \frac{1}{2} + \dots + \frac{1}{N} - \ln N \rightarrow r \text{ 欧拉常数} \\ p < 1 \end{cases}$$

$$p=2 \quad \sum \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum \frac{1}{n^{2p}} = \frac{\pi^{2p}}{2^{2p}}, \quad \sum \frac{1}{n^{2p-1}} = \frac{\pi^{2p-1}}{2^{2p-1}}$$

正项级数

比较判别法 $\exists N > 0, \text{ s.t. } \forall n > N, 0 \leq U_n \leq C V_n \quad \sum V_n \text{ 收} \rightarrow \sum U_n \text{ 收}; \sum U_n \text{ 发} \rightarrow \sum V_n \text{ 发}。$

$$\text{极限形式: } \lim_{n \rightarrow \infty} \frac{U_n}{V_n} = l$$

① $0 < l < +\infty$: 同敛散

② $l=0$ ($U_n < C V_n$) $\sum U_n$ 收, $\sum U_n$ 发; $\sum U_n$ 发, $\sum V_n$ 发

③ $l=+\infty$ ($U_n > C V_n$) $\checkmark$

$$\sum U_n \text{ 收敛} \iff S_N = \sum_{n=1}^N U_n \text{ 有上界}$$

S_n 单调有上界

比值审敛法

$$\frac{U_{n+1}}{U_n} \leq q < 1, U_n \text{ 收}; q > 1, U_n \text{ 发}$$

$$U_{n+1} \leq q^n U_1, \sum q^n U_1 \text{ 收敛}$$

$$\text{极限形式: } \lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = l \quad \textcircled{1} l < 1 \text{ 收} \quad \textcircled{2} l > 1 \text{ 发} \quad \textcircled{3} l = 1 \text{ UB}$$

根值审敛法

$$\sqrt[n]{U_n} \leq q < 1, \text{收}; q > 1, \text{发}$$

$$U_n \leq q^n$$

$$\text{极限形式 } \lim_{n \rightarrow \infty} \sqrt[n]{U_n} = l \quad \checkmark$$

积分审敛法

a_n 单减, f 在 $(1, +\infty)$ 连续单减的 $f(x)$, s.t. $f(n) = a_n (n \in \mathbb{Z}^+)$

$\lim_{n \rightarrow \infty} \int_1^n f(x) dx$ 与 $\sum a_n$ 同敛散

e.g. $\sum a_n = S, \sum \sqrt[n]{a_n a_{n+1}} \leq \sum \frac{a_n + a_{n+1}}{2}$ 收

$$\sum \frac{a_n}{1+a_n}$$

$$\sum a_n \text{ 收} \rightarrow \exists N > 0, \forall n > N, 0 < a_n < \frac{1}{2} \quad \text{取 } t = \frac{1}{2}$$

$$\forall n > N, \frac{1}{1+a_n} < 1 \rightarrow \frac{a_n}{1+a_n} < a_n \quad \sum 2a_n \text{ 收}, \sum \frac{a_n}{1+a_n} \text{ 收}.$$

交错级数 $\sum_{n=1}^{\infty} (-1)^n a_n$

Leibniz 判别法: a_n 单减 $\rightarrow 0, \sum_{n=1}^{\infty} (-1)^n a_n$ 收敛. $\sum \frac{1}{n}$ 发散, $\sum (-1)^n \frac{1}{n}$ 收敛

一般级数

$\sum |a_n|$ 收 $\rightarrow \sum a_n$ 收. 绝对、条件收敛. $\sum |a_n| \rightarrow$ 利用正项系数审敛, 判断绝对收敛性

极限审敛 $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = l$. $l < 1$ 绝对; $l > 1$ 发; $l = 1$ UB.

函数项级数 $\sum_{n=1}^{\infty} U_n(x) = S(x), x \in I$ 收敛域

泰勒公式: $e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{扔掉余项. } \triangle$$

$$\sum_{n=1}^{\infty} x^n \text{ 例 } \textcircled{1} |x| < 1, S_n = \frac{1-x^{n+1}}{1-x} \xrightarrow{n \rightarrow \infty} \frac{1}{1-x}$$

$$\textcircled{2} |x| > 1, \text{ 发散}$$

$$\text{即 } I = (-1, 1), \text{ 收敛域}$$

$$\textcircled{3} |x| = 1, x = 1, S_n = n \rightarrow +\infty; x = -1, \sum (-1)^n \text{ 发}.$$

$\sum a_n(x-a)^n$ 关于 $x-a$ 的幂级数, $I = (a-R, a+R)$, R 为收敛半径, I 为收敛域

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho = \lim_{n \rightarrow \infty} \left| \frac{1}{a_n} \right|, R = \frac{1}{\rho} \quad \text{计算 } I, \text{ 单独判断边界 } (a-R, a+R)$$

e.g. $\sum_{n=0}^{\infty} \frac{x^n}{n!}, a_n = \frac{1}{n!}, \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 = \rho, R = +\infty, I = (-\infty, +\infty) = \mathbb{R}, \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x \quad \sum \frac{x^n}{n!} = e^x$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \quad \rho = 1, R = 1, x = 1, \sum (-1)^n \frac{1}{n!} \text{ 收}; x = -1, \sum \frac{1}{n!} \text{ 发}. I = (-1, 1)$$

缺点 e.g. $\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2^n} \neq \sum a_n x^n$

$$\forall x_0 \in \mathbb{R}, U_n = \frac{x_0^{2n-1}}{2^n}, \text{ 常项级数 } \left| \frac{U_{n+1}}{U_n} \right| \rightarrow \frac{x_0^2}{2} \quad \frac{x_0^2}{2} < 1, |x_0| < \sqrt{2}, \sum U_n \text{ 收}; |x_0| > \sqrt{2}, \sum U_n \text{ 发}$$

$$\text{即 } R = \sqrt{2}, x = \sqrt{2}, \sum \frac{1}{2^n} = +\infty \text{ 发}; x = -\sqrt{2}, \text{ 发}. I = (-\sqrt{2}, \sqrt{2}) \quad \text{幂级数的收敛域一定对称}$$

性质 $\sum_{n=0}^{\infty} a_n x^n = S(x)$

$$1. S(x) \text{ 在 } I \text{ 内关于 } x \text{ 连续: } \lim_{n \rightarrow \infty} S(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k x^k$$

$$2. S(x) \text{ 可以在 } I \text{ 内逐项积分、逐项求导. } S'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}, \int_0^x S(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \left. \vphantom{\int_0^x S(t) dt} \right] \lim, \int, \text{ 导与 } \sum \text{ 互换}$$

e.g. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2^{n-1}}, |x| < 1$

$$S'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n-2} = 1 - x^2 + x^4 - x^6 + \dots = \frac{1}{1+x^2}$$

$$\int_0^x \frac{1}{1+t^2} dt = \int_0^x S'(t) dt = S(x) - S(0) = S(x) = \arctan x \quad (|x| < 1)$$

$$(1) f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots \quad \text{Taylor}$$

$$(2) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad (-\infty < x < +\infty)$$

$$(3) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \dots \quad (-\infty < x < +\infty)$$

$$(4) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + \dots \quad (-\infty < x < +\infty)$$

$$\ln(1+x)$$

$$(5) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots \quad (-1 < x \leq 1)$$

$$(6) (1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{1 \cdot 2}x^2 + \dots + \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots \quad (-1 < x < 1)$$

$$(7) \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots \quad (-1 < x \leq 1)$$

$$\arctan x$$

$$(8) \sin^{-1} x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \dots + \frac{1 \cdot 3 \dots (2n-1)x^{2n+1}}{2 \cdot 4 \dots 2n(2n+1)} + \dots \quad (-1 < x \leq 1) \quad (2.91)$$

$$\text{e.g. 证 } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots$$

$$S(x) = \ln(1+x)$$

$$S'(x) = \frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots, |x| < 1$$

$$\int_0^x S'(t) dt = \int_0^x 1 - t + t^2 - \dots = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n dt$$

$$S(x) - S(0) = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt$$

△

计算定积分 无法写出原函数

$$\int_0^1 \frac{\ln(1+x)}{x} dx = \int_0^1 (1 - \frac{x}{2} + \frac{x^2}{3} - \dots) dx = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}$$

幂级数展开

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$f(x) = \frac{1}{2+x-x^2} = \frac{1}{3} \left(\frac{1}{1+x} + \frac{1}{1-x} \right) = \frac{1}{3} \cdot \frac{1}{1+x} + \frac{1}{3} \cdot \frac{1}{1-x} \quad \text{无穷等比数列求和}$$

$$= \frac{1}{3} \cdot \sum_{n=0}^{\infty} (-1)^n x^n + \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{x}{1}\right)^n \quad (|x| < 1 \quad (I_1: |x| < 1, I_2: |x| < 1))$$

$$f(x) = \frac{1}{x^2-3x-4} \text{ 展开为关于 } (x-1) \text{ 的幂级数}$$

$$x-1=t \quad f(1+t) = \frac{1}{t^2-5t-6} = -\frac{1}{5} \left(\frac{1}{2+t} + \frac{1}{3-t} \right)$$

幂级数乘幂级数

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \cos x dx \quad \text{求 } \sum I_n$$

$$= \frac{1}{n+1} \left(\frac{\pi}{2} \right)^{n+1}$$

$$\sum I_n = S\left(\frac{\pi}{2}\right), \quad p=1, \quad R=1 \quad \frac{\pi}{2} \in I, \quad S(x) = \sum_{n=1}^{\infty} \frac{1}{n+1} x^{n+1}, \quad \text{求 } S(x)$$

$$S'(x) = \sum x^n = \frac{1}{1-x} \quad (|x| < 1)$$

$$S(x) - S(0) = \int_0^x \frac{1}{1-t} dt = -\ln|1-x| \Rightarrow S(x) = -\ln|1-x| \Rightarrow \text{LHS} = S\left(\frac{\pi}{2}\right) = -\ln\left(1-\frac{\pi}{2}\right) = \ln\left(2+\sqrt{2}\right)$$

傅里叶级数

$$\text{在 } [x, \pi] \text{ 上连续光滑的 } f(x), \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum C_n e^{inx}, \quad e^{inx} = \cos nx + i \sin nx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

若 f 为周期函数，可平移

$$\text{奇: } a_n = 0$$

$$\text{偶: } b_n = 0$$

在 $[-l, l]$ 上, $f(x) = \frac{a_0}{2} + \sum (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}) = \sum C_n e^{-\frac{i n \pi x}{l}}$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

✓

$$C_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{i n \pi x}{l}} dx$$

正交性: 系数推导

$$\int_{-l}^l \sin m x \sin n x dx = \int_{-l}^l \cos m x \cos n x dx = \begin{cases} 0, m \neq n \\ \pi, m = n \end{cases}, \quad \int_{-l}^l \sin m x \cos n x dx = 0$$

$$\text{DEF } \langle f, g \rangle = \frac{1}{\pi} \int_{-l}^l f g dx$$

取 $1, \cos x, \cos 2x, \dots, \sin x, \sin 2x, \dots$ 为 basis

因此 $m=n$ 时, f, g 视为“平行”, 内积为模; $m \neq n$, f 与 g “垂直”, 内积为 0

有 $X = A X_0 + B X_1 + C X_2 + D X_3$, X_0, X_1, X_2, X_3 两两正交

则有 $X \cdot X_0 = A |X_0|^2 + 0$, 得 A 。 B, C, D 同理。

$$\forall k \in \mathbb{Z}^+, f(x) \cos k x = \frac{a_0}{2} \cos k x + \sum_{n=1}^{\infty} (a_n \cos n x \cos k x + b_n \sin n x \cos k x)$$

$$\int_{-l}^l f(x) \cos k x dx = \frac{a_0}{2} \int_{-l}^l \cos k x dx + \sum_{n=1}^{\infty} a_n \int_{-l}^l \cos n x \cos k x dx + \sum_{n=1}^{\infty} b_n \int_{-l}^l \sin n x \cos k x dx$$

$$\text{由正交性, } = 0 + a_k \int_{-l}^l \cos k x \cos k x dx = \pi a_k$$

$$a_k = \frac{1}{\pi} \int_{-l}^l f(x) \cos k x dx, \quad \forall k \in \mathbb{Z}^+ \quad n \text{ 代 } k$$

Parseval 恒等式

$$f(x) = \dots \quad \text{展开}$$

$$\frac{1}{\pi} \int_{-l}^l |f(x)|^2 dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \sum_{n=-\infty}^{\infty} |C_n|^2 \quad \langle f, f \rangle$$

交错相减求得部分和: $S_N = \sum_{n=1}^N a_n$, 则 $\sum a_n = \lim_{N \rightarrow +\infty} S_N$

作业4

$$\begin{aligned} 1. & \int 2^{2x+3} dx \\ &= \frac{1}{2} \int 2^{2x+3} d(2x+3) \\ &= \frac{1}{2} \frac{2^{2x+3}}{\ln 2} = \frac{1}{\ln 2} \cdot 2^{2x+2} + C \end{aligned}$$

$$\begin{aligned} 2. & \int (x - \frac{1}{\sqrt{x}})^2 dx \\ &= \int (x^2 - 2\sqrt{x} + \frac{1}{x}) dx \\ &= \frac{1}{3}x^3 - \frac{1}{\sqrt{x}} - \frac{1}{x^2} + C \end{aligned}$$

$$\begin{aligned} 3. & \int \sqrt{x^2 + a^2} dx \quad x = a \tan t \\ &= a^2 \int \sec^3 t dt \quad dx = a \cdot \sec^2 t \\ &= a^2 \int (\sec t \cdot (\tan t)') dt \\ &= a^2 (\sec t \tan t - \int \tan^2 \sec t dt) \\ &= a^2 (\sec t \tan t - \int \sec t (\sec^2 t - 1) dt) \\ &= a^2 \sec t \tan t - a^2 \int \sec^3 t dt + a^2 \int \sec t dt \end{aligned}$$

$$\text{取 } I = \int \sec^3 t dt$$

$$I = \sec t \cdot \tan t - I + \int \sec t dt$$

$$\frac{\sqrt{x^2+a^2}}{\frac{1}{a}} \cdot \frac{x}{a}$$

$$I = \frac{1}{2} (\sec t \tan t + \int \sec t dt)$$

$$= \frac{1}{2} [\sec t \tan t + \ln |\sec t + \tan t|] + C$$

$$\text{原式} = \frac{a^2}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2+a^2}}{a} + \ln \frac{\sqrt{x^2+a^2} + x}{a} + C$$

作业 6

$$\begin{aligned}
 1. \quad V &= \int_0^\pi \pi \cdot \sin^2 x \, dx = \pi \int_0^\pi \sin^2 x \, dx \\
 &= 2\pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \pi \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx \\
 &= \pi \left[\frac{x}{2} - \frac{1}{2} (-\sin 2x) \right]_0^{\frac{\pi}{2}} \\
 &= \frac{1}{2} \pi^2
 \end{aligned}$$

$$\begin{aligned}
 2. \quad &\begin{cases} x = a(t - \sin t) & 0 \leq t \leq 2\pi \\ y = a(1 - \cos t) & 0 \leq t - \sin t \leq \frac{2\pi}{a} \end{cases} \\
 S &= [a(1 - \cos t)]^2 \cdot \pi \cdot dx \\
 &= a^2 (1 - \cos t)^2 \cdot \pi \cdot a(1 - \cos t) dt \\
 &= \pi a^3 (1 - \cos t)^3 dt \\
 V &= \int_0^{2\pi} S \cdot dt = \int
 \end{aligned}$$

作业 8

$$1. 0 \leq \frac{\sin 2^n}{2^n}, a_n = \frac{\sin 2^n}{2^n} \xrightarrow{t=2^n} \frac{\sin t}{t}$$

$$|a_n| < \frac{1}{2^n}, \sum |a_n| < \sum \frac{1}{2^n}, \text{收}$$

$$\frac{1}{2^{n+1}} / \frac{1}{2^n} = \frac{1}{2} < 1$$

$$3. a_n = \frac{(-1)^{n+1} n^2}{2n^2+1} \quad \lim_{n \rightarrow \infty} a_n = \begin{cases} n=2k, -\frac{1}{2} \\ n=2k+1, \frac{1}{2} \end{cases}, a_n \text{ 极限不存在} \rightarrow \sum a_n \text{ 发散}$$

$$|a_n| = \frac{1}{2 + \frac{1}{n^2}} < \frac{1}{2}$$

2. 正项.

$$0. a_n = \frac{2n-1}{2^n}, \frac{a_{n+1}}{a_n} = \frac{1}{2} \cdot \frac{2(n+1)-1}{2n-1} = \frac{1}{2} \cdot \frac{2n+1}{2n-1} \rightarrow \frac{1}{2} < 1$$

$$S_N = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \dots + \frac{2N-1}{2^N}$$

$$\frac{1}{2} S_N = \frac{1}{2^2} + \frac{3}{2^3} + \frac{5}{2^4} + \dots + \frac{2N-1}{2^{N+1}}$$

$$\frac{1}{2} S_N = \frac{1}{2} + 2 \left(\frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^N} \right) - \frac{2N-1}{2^{N+1}}$$

$$= \frac{1}{2} + \left(\frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^{N+1}} \right) - \frac{2N-1}{2^{N+1}}$$

$$S_N = 1 + 2 \sum_{i=1}^N \frac{1}{2^i} - \frac{2N-1}{2^{N+1}}$$

$$\rightarrow 1 + \frac{1}{1-2} - 0 = 3.$$

$$2. a_n = \frac{1}{(5n-4)(5n+1)} = \left(\frac{1}{5n-4} - \frac{1}{5n+1} \right) \cdot \frac{1}{5}$$

$$\frac{a_{n+1}}{a_n} = \frac{(5n-4)(5n+1)}{(5n+1)(5n+6)} \rightarrow 1 \text{ UB.}$$

先求 S_N

$$S_N = \frac{1}{5} \left(1 - \frac{1}{5N+1} \right) \rightarrow \frac{1}{5}$$

3. $a_n = \left(\frac{n}{2n+1} \right)^n$

$$\sqrt[n]{a_n} = \frac{n}{2n+1} \rightarrow \frac{1}{2} < 1$$

$$a_n = \frac{n!}{n^n}$$

$$\frac{a_{n+1}}{a_n} = (n+1) \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1} \right)^n < 1^n = 1 \quad \Delta$$

$$= \left(\frac{1}{1+\frac{1}{n}} \right)^n \Rightarrow \frac{1}{e} < 1$$

4. $|a_n| = \left| \frac{\sin n}{n!} \right| < \frac{1}{n!} = b_n$ 收

$$\frac{b_{n+1}}{b_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} < 1 \text{ 收}$$

5. $S_N = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{N} - \ln N$ S_N 收敛 $\rightarrow$ 级数收敛

$$S_N > \ln(1+1) + \ln(1+\frac{1}{2}) + \dots + \ln(1+\frac{1}{N}) - \ln N$$

$$= \ln 2 + \ln \frac{3}{2} + \ln \frac{4}{3} + \dots + \ln \frac{N+1}{N} - \ln N$$

$$= \ln 2 + \ln 3 + \ln 4 + \dots + \ln N + \ln(N+1) - \ln N = \ln \frac{N+1}{N} > 0 \text{ 有下界.}$$

$$S_{N+1} - S_N = \frac{1}{N+1} - \ln(N+1) + \ln N \geq 0$$

$$= \frac{1}{N+1} - \ln(1+\frac{1}{N}) < 0 \text{ 单调口}$$

$$\frac{1}{n+1} < \ln(1+\frac{1}{n}) < \frac{1}{n}$$

6. $f(x) = |x|, x \in [-\pi, \pi]$ 偶函数 $\rightarrow b_n = 0$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{1}{\pi} \int_{-\pi}^0 (-x) \cos nx dx + \frac{1}{\pi} \int_0^{\pi} x \cos nx dx$$

$$-\frac{1}{\pi} \int_{-\pi}^0 x \cos nx dx = -\frac{1}{\pi\pi} \int_{-\pi}^0 x d \sin nx = -\frac{1}{\pi\pi} \left[x \sin nx \Big|_{-\pi}^0 - \int_{-\pi}^0 \sin nx dx \right] = -\frac{1}{\pi\pi} \left(-\frac{1}{n} \right) \cos nx \Big|_{-\pi}^0$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^0 (-x) dx + \frac{1}{\pi} \int_0^{\pi} x dx = \frac{1}{\pi} \left(\frac{1}{2} x^2 \Big|_0^{\pi} - \frac{1}{2} x^2 \Big|_{-\pi}^0 \right) &= \frac{-\frac{1}{\pi\pi} (1 - \cos n\pi)}{\frac{1}{\pi n^2} ((-1)^n - 1)} \\ &= \frac{1}{\pi} \left[\frac{1}{2} \pi^2 - (0 - \frac{1}{2} \pi^2) \right] &= \frac{1}{\pi} \cdot \pi^2 = \pi \end{aligned}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2} [(-1)^n - 1] \cdot \cos nx = \frac{\pi}{2} +$$

