$Ax = b$

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ c_1 & c_2 & c_3 \\ \vdots & \vdots & \vdots \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{matrix} a \times c_1 \\ + \\ b \times c_2 \\ + \\ c \times c_3 \end{matrix}$$

, Matrix · Column = Column , Linear combinition of columns

Right, do column operations

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} \text{---} R_1 \text{---} \\ R_2 \\ R_3 \end{bmatrix} = aR_1 + bR_2 + cR_3,$$

Row · Matrix = Row, Linear combinition of rows

Left, do row operations

解N元N次方程组：高斯消元 Elimination ⟶ 矩阵表达：

$(Ax = b)$

$AB = C$, $C_{ij} = (\text{Row}_i \text{ of } A) \cdot (\text{Col}_j \text{ of } B)$

$$\underset{A}{\begin{bmatrix} & \end{bmatrix}} \underset{B}{\begin{bmatrix} B_1 & | & | & | & B_n \end{bmatrix}} = \underset{C}{\begin{bmatrix} C_1 & | & | & | & C_n \end{bmatrix}}$$

$C_1 = A \cdot B_1$, $C_2 = A \cdot B_2$ …… C_i is a linear combination of A's cols.

Columns of C are combination of columns of A.

Chapter 3 Vector spaces and Subspaces

$Ax = b$, if b is in A's column space, 有解。

x做A的列的线性组合

$$\begin{array}{c} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \\ A_1 \; A_2 \; A_3 \end{array}$$

丢弃A_3，仅$A_1 A_2$也会得到相同的列空间 $(A_3 = A_1 + A_2)$

A_1, A_2是 pivot column

Nullspace $Ax = 0$ 的所有解x. $b = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad x \text{ in } R^3$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ c \\ -c \end{bmatrix} \cdots \Rightarrow c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$x_1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$Av = 0$, $Aw = 0$, $A(av + bw) = 0$ ∴ Indeed a vector space

if $b \neq 0$, X_s 不会形成 vector space

Form a subspace : ① some vectors, take linear combinations.

② $Ax = 0$, X_s form a subspace, which is a Nullspace

Solving $Ax=0$

① elimination

$$A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}$$ $Ax=0$

column 2 is …

Nullspace unchange 解

Vector space change

Pivot

$$\longrightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&2&4\end{bmatrix}$$

$$\longrightarrow\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}=U$$ 2 pivots $\Longrightarrow$ The Rank of A is 2 pivot的数量

$Ux=0$

pivot column 2 pivot

free column x_2, x_4 的解为任意值

2 free variables

② special solutions

i) Let $x_2=1,\ x_4=0$

$$x=c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}$$ $x_2=c_1,\ x_4=0$

ii) Let $x_2=0\quad x_4=1$

$$x=c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$$ $x_4=c_2,\ x_2=0$

So, $x=c_1\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\0\\-2\\1\end{bmatrix}$, done.

$(x_2,x_4)=(c_1,c_2)\rightarrow$ Every number

Matrix $m\times n$, Rank r, free vars $=n-r$

Reduced row echelon form

$$U=\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&0&-2\\0&0&2&4\\0&0&0&0\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&0&-2\\0&0&1&2\\0&0&0&0\end{bmatrix}=R$$, Let pivots to be 1 做②方便

zeros both above and below the pivots

$Rx=0$

pivot / identity part: $\begin{bmatrix}1&0\\0&1\end{bmatrix}=I$

free part: $\begin{bmatrix}2&-2\\0&2\end{bmatrix}=F$

$R=\begin{bmatrix}I&F\\0&0\end{bmatrix}$ to solve $Rx=0$, so xs form a Nullspace N, s.t. $RN=0$, $N=\begin{bmatrix}-F\\I\end{bmatrix}$

$$[I\ \ F]\begin{bmatrix}x_{pivot}\\x_{free}\end{bmatrix}=0\longrightarrow x_{pivot}=-F\cdot x_{free}$$

Solving $Ax=b$

$[A\ b]$ Augmented Matrix

$$\begin{bmatrix}1&2&2&2&b_1\\2&4&6&8&b_2\\3&6&8&10&b_3\end{bmatrix}\longrightarrow\begin{bmatrix}1&2&2&2&b_1\\0&0&2&4&b_2-2b_1\\0&0&0&0&b_3-b_2-b_1\end{bmatrix}$$ $\Longrightarrow 0=b_3-b_2-b_1$ to have solutions

$R_3-R_1-R_2=$ zero rows $\longrightarrow$ must 0

Solvability of $Ax=b$: b is in the $C(A)$ column space of A

$\Longleftrightarrow$ If a combination of rows of A gives zero rows, then the same comb of entries of b must give 0.

Let's find the solution(s)

① Set free vars (x_2,x_4) to zero, give $x_1=-2$, $x_3=\frac{3}{2}$, $\begin{bmatrix}-2\\0\\3/2\\0\end{bmatrix}=x_{particular}$

② Solve $Ax = 0 \Longrightarrow X_{null}$ $\qquad AX_{null} = 0, \ AX_p = b$

$\Downarrow$

③ $X = X_{null} + X_{particular}$ $\qquad A(X_p + X_N) = b$ ✓

Why it gives all the solutions?

$m \times n$ Matrix with r Rank $\qquad r \le n, r \le m$

Full Rank

① full column rank, $r = n \longrightarrow n$ pivot vars, no free vars $\longrightarrow X_N = 0$

for $Ax = b$, $X = X_p$ 唯一解/无解

② full row rank, $r = m \longrightarrow m$ pivot vars, $n-m$ free vars

Back to $Ax = b$

i) $r = m = n$ Full Rank Invertible Matrix A $\qquad$ 1 solution, $R = I$

ii) $r = m < n$ $\quad R = [I \ F]$ specially. ∞ solutions

iii) $r = n < m$ $\quad R = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\quad 0/1$ solution

iv) $r < m, r < n$ $\qquad 0/\infty$ solutions

Vector independence

$C_1V_1 + C_2V_2 + \cdots + C_nV_n \ne 0$, for $\forall C_1 \cdots C_n$, $C_1 \cdots C_n$ 不全为0 $\quad$ If there is a zero vector, they can't be independent

$A = [V_1, V_2, \cdots V_n]$, $Ax = 0 \longrightarrow x = 0 \Longleftrightarrow V_1, \cdots V_n$ are independent

$\Longleftrightarrow$ A's nullspace only 0

$\Longleftrightarrow r = n$, no free vars

vectors span a space

take all linear combinations of these vectors

Basis for a space: a sequence of vectors: ① span the space ② independent

Dimension for a space with N demension, all basis have N vectors.

For R^n, $[V_1 \cdots V_n]_{n \times n}$ is invertible $\Longleftrightarrow V_1 \cdots V_n$ is the basis of R^n

rank(A) = # pivot cols = dimension of C(A)

How to find basis if we know DIM?

For Null space?

$\dim N(A) = \#\text{free} = n - r$.

4 Subspaces

$C(A)$	$\mathbb{R}^m$	Basis: pivot cols,	r dimension
$N(A)$	$\mathbb{R}^n$	special solutions	$n-r$ #free variables ? $\dim(N(A)) + \dim(R(A)) = \#\text{cols}$
$R(A) = C(A^T)$	$\mathbb{R}^n$	In R Matrix	r, $\text{Rank}(A) = \text{Rank}(A^T) = \dim(C(A)) = \dim(R(A))$
$N(A^T)$	$\mathbb{R}^m$		$m-r$

also called left null space

$$[A_{m\times n} \; I_{m\times m}] \longrightarrow [R_{m\times n} \; E_{m\times n}] \iff EA = R, \quad \text{sp. } E = A^{-1}, \; R = I$$